University of Kelaniya- Sri Lanka
Faculty of Science
Centre for Distance & Continuing Education
Bachelor of Science (General) Degree Examination-External
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PURE MATHEMATICS | PMAT 36602 (Repeat)- Abstract Algebra

No. of Questions: Five (05) **No. of Pages:** Two (02) **Time:** Two (02) hours

Answer only FOUR (04) questions.

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1. (a) If G is a group under the operation $*$, then prove that the **identity** of G is unique.

   (b) Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$.

   i. Prove that G is a group under the usual multiplication.

   ii. Prove that G is not a group under addition.

   (c) If x and g are elements of the group G, prove in the usual notation that $|x| = |g^{-1}xg|$.

2. (a) i. Prove that if H and K are subgroups of a group G, then their intersection $H \cap K$ is also a **subgroup** of G.

   ii. Let $\mathbb{R}'$ denote the set of real numbers whose square is a rational number ($\mathbb{R}' = \{r \in \mathbb{R} \mid r^2 \in \mathbb{Q}\}$). Show that $\mathbb{R}'$ is **not a subgroup** of $\mathbb{R}$ under addition.

   (b) i. State **Lagrange's theorem**.

   ii. Prove that if G is a finite group and $x \in G$, then the order of x divides the order of G.

   (c) Prove that if G is an abelian group, then every subgroup of G is **normal**.

3. (a) Let G be an abelian group. Define $\psi : G \longrightarrow G$ by $\psi(g) = g^{-1}$. Prove that ψ is a **homomorphism**.

   (b) Let G and H be groups and let $\phi : G \longrightarrow H$ be a homomorphism. Prove that, $\phi(e_G) = e_H$, where e_G and e_H denote the identities of G and H respectively.

   (c) i. If $\phi : G \longrightarrow H$ is an **isomorphism**, prove that G is abelian if and only if H is abelian.

   ii. If $\phi : G \longrightarrow H$ is a homomorphism, what additional conditions on ϕ (if any) are sufficient to ensure that if G is abelian, then so is H?

Continued.

4. (a) i. Define a **zero divisor** and a **unit** in a ring R.

 ii. Find all zero divisors and units (if any) in the **ring $\mathbb{Z}$ of integers**.

 (b) Let $M_2(\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}\}$ be the set of $n \times n$ matrices with entries from $\mathbb{Z}$.

 i. Prove that $M_2(\mathbb{Z})$ is a ring, with identity 1 (in the usual notation), under the usual addition and multiplication of matrices.

 ii. Find the set of units in $M_2(\mathbb{Z})$.

 (c) Let R be a ring. Assume a, b and c are elements R where a is not a **zero divisor**. If $ab = ac$, then prove that either $a = 0$ or $b = c$.

5. (a) Consider the ring $\mathbb{Q}[i] = \{a + ib \mid a, b \in \mathbb{Q}\}$ under the usual addition and multiplication of complex numbers. Prove that $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}\}$ is a **subring** of $\mathbb{Q}[i]$.

 (b) i. Show that $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $\phi(x) = x^2$ is **not a ring homomorphism**.

 ii. Let $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_5$ be defined by $\phi(x) = x \pmod{n}$. Prove that ϕ is a **ring homomorphism**.

 (c) Determine the **characteristics** of the the rings, $\mathbb{Z}$ and $\mathbb{Z}_{10}$.

....**END OF EXAMINATION**..