University of Kelaniya - Sri Lanka
Center for Distance & Continuing Education
Bachelor of Science(General) External
Third year second semester examination - 2019 (2025 March)
(New Syllabus)
Faculty of Science

Pure Mathematics
PMAT 37612 -Theory of Riemann Integration

No.of Questions: Five(05) **No.of Pages: Three(03)** **Time: Two(2)hrs**
Answer Four(04) Questions Only

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1. (a) Let f be a bounded and positive function defined on $[a, b]$. Let P be a partition of $[a, b]$. In the usual notations, define upper Darboux sum $U(f, P)$, lower Darboux sum $L(f, P)$, upper Darboux integral $U(f)$ and lower Darboux integral $L(f)$.

[20 marks]

(b) The function $f : [1, 3] \to \mathbb{R}$ is defined by $f(x) = x^2 + 3$. Let P be a partition of $[1, 3]$ with equally spaced n sub intervals.

(i) Calculate $U(f, P)$, $L(f, P)$, $U(f)$ and $L(f)$ on the interval $[1, 3]$.

[50 marks]

(ii) Is f Darboux integrable over $[1, 3]$? Justify your answer.

[05 marks]

(c) Let f be a bounded function on $[a, b]$. If P be a partition of $[a, b]$ then prove $U(f, P)$, $L(f, P)$ are bounded and $U(f, P) \geq L(f, P)$ for all partition P.

[25 marks]

2. (a) Prove that a bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

[30 marks]

(b) Let f be a bounded Riemann integrable function on $[a, b]$. Suppose (S_n) is a sequence of Riemann sums, with corresponding partitions p_n, satisfying $\lim_{n \to \infty} ||p_n|| = 0$. Then prove the sequence S_n converges to $\int_a^b f$.

[20 marks]

Continued...

(c) Prove the following properties of Riemann integrals.

(i) If f and g are integrable on $[a,b]$ and if $f(x) \leq g(x)$ for x in $[a,b]$, then $\int_a^b f \leq \int_a^b g$.

[25 marks]

(ii) If g is a continuous nonnegative function on $[a,b]$ and if $\int_a^b g = 0$, then g is identically 0 on $[a,b]$.

[25 marks]

3. (a) (i) State the Dominated Convergence theorem and Monotone Convergence theorem, in the usual notations.

[20 marks]

(ii) Justifying your calculation, compute the following limits.

(A) $\lim_{n\to\infty} \int_a^b \left(1+\frac{x}{4n}\right)^n dx$ where $a > 0$ (B) $\lim_{n\to\infty} \int_3^4 \left(\frac{n^2 x \sin^2 x}{1+(nx)^2}\right) dx$ where $n \geq 1$.

[40 marks]

(b) State the First Fundamental Theorem of Calculus. Hence find $\int_0^5 f\,dx$, where

$$f(x) = \begin{cases} \sqrt{x}\sec^2 x - \dfrac{\tan x}{2\sqrt{x}} & ; \quad 0 < x \leq 5 \\ 0 & ; \quad x = 0 \end{cases}.$$

[40 marks]

4. (a) (i) Explain the two types of improper integrals.

[10 marks]

(ii) Evaluate the following improper integrals.

(A) $\int_0^4 \frac{dx}{(x^2-x-2)}$ (B) $\int_0^\infty x e^{-x^2}\,dx$

[50 marks]

Continued...

(b) (i) Define the convergence and divergence of improper integral.

[10 marks]

(ii) What is meant by saying that the improper integral is absolutely convergent and conditionally convergent.

[20 marks]

(iii) Determine the convergence of $\int_1^{\infty} \frac{\cos x}{x^2}\, dx$.

[10 marks]

5. (a) Show that the improper integral $\int_a^b \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges for $p \geq 1$.

[20 marks]

(b) State and prove the Comparison test of Type II.

[20 marks]

(c) Examine whether the following improper integrals are convergent or divergent. Justify your answers.

$$\text{(i)} \int_1^{\infty} \frac{x^2+3x}{\sqrt{x^5+1}}\, dx \quad \text{(ii)} \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{\tan x}}$$

$$\text{(iii)} \int_0^{\sqrt{2}} \frac{dx}{x\sqrt{2-x^2}} \quad \text{(iv)} \int_3^{\infty} \frac{1+\cos^2 x}{\sqrt{x}(2-\sin^4 x)}\, dx$$

[60 marks]

— End of Examination —