University of Kelaniya- Sri Lanka
Faculty of Science
Centre for Distance & Continuing Education
Bachelor of Science (General) Degree Examination- External
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PURE MATHEMATICS | PMAT 37622- Mathematical Methods
No. of Questions: Five (05) **No. of Pages:** Two (02) **Time:** Two (02) hours
Answer only FOUR (04) questions.

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1. (a) Using the **Heaviside step function**, determine the Laplace transform of the sine function $S(t)$ switched on at time, $t = 2$.

$$S(t) = \begin{cases} \sin t, & t \geq 2 \\ 0, & t < 2 \end{cases} .$$

(b) i. If $\mathcal{L}\{f(t)\} = F(s)$ then prove that

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$$

and in general

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

.

ii. Determine the Laplace transform of the function $tS(t)$, where $S(t)$ is the function given in part (a) above.

2. (a) Using the concept of **convolution**, find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{6s}{(s^2+1)^2}\right\}.$$

(b) Use Laplace transform techniques to find the solution to the second order differential equation

$$\frac{d^2y}{dx} + 5\frac{dy}{dx} + 6y = 2e^{-x}, \qquad x \geq 0,$$

subject to the conditions $y(0) = 1$ and $y'(0) = 0$.

3. (a) Consider the function $f(t) = e^{-2t}$. By considering the Fourier cosine and sine transforms of the function $f(t)$, show that

$$F_c(\omega) + iF_s(\omega) = \frac{2 + i\omega}{4 + \omega^2},$$

where $F_c(\omega)$ and $F_s(\omega)$ are Fourier cosine and Fourier sine transform of $f(t)$ respectively.

Continued.

Hence prove that,

i.

$$\int_0^\infty \frac{\cos(kx)}{4+x^2}\, dx = \frac{\pi}{2a} e^{-2k} \quad \text{and,}$$

ii.

$$\int_0^\infty \frac{x\sin(kx)}{4+x^2}\, dx = \frac{\pi}{2} e^{-2k}.$$

(b) Consider the following periodic function.

$$f(x) = |x| + 1, \quad -3 \le x \le 3, \text{ with } f(x) = f(x+6).$$

Using the properties of even and odd functions and the definition of a Fourier series. determine the Fourier Series of $f(x)$.

4. (a) Consider the function $f(x) = 3x,\ 0 < x < 2$.

   i. Expand $f(x)$ given above in a half-range **Cosine Series**.

   ii. Deduce that $\pi^2/8 = 1 + 1/3^2 + 1/5^2 + ...$

   (b) Find the solution of

$$y^2 \frac{\partial u}{\partial y} + x^3 \frac{\partial u}{\partial x} = 0$$

   given that $u(0, y) = 20e^{-3/y}$.

5. Consider the following one dimensional wave equation and the corresponding boundary and initial conditions, in the usual notation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Boundary conditions: $u(o, t) = u(L, t) = 0$ for all $t \ge 0$.
Initial conditions: $u(x, 0) = f(x),\quad u_t(x, 0) = g(x)$ for all $0 \le x \le L$.

   (a) Using the method of separating variables, setting $u(x, t) = F(x)G(t)$ show that

$$F'' - kF = 0 \quad \text{and} \quad G'' - c^2kG = 0$$

   where k is the separation constant.

   (b) Show that k must be negative.

   (c) In the usual notation, show that the finitely many solutions of $F(x)$ is given by $F_n(x) = \sin \frac{n\pi x}{L}$.

   (d) Find the solution of the entire problem.