University of Kelaniya- Sri Lanka
Faculty of Science
Centre for Distance & Continuing Education
Bachelor of Science (General) Degree Examination-External
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PURE MATHEMATICS | PMAT 36602- Abstract Algebra

**No. of Questions:** Five (05) **No. of Pages:** Two (02) **Time:** Two (02) hours

**Answer only FOUR (04) questions.**

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1. (a) If $G$ is a group under the operation $*$, then prove that for each $a \in G$, $a^{-1}$ is uniquely determined.

   (b) Let $G = \{\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \mid a, b \in \mathbb{R}, a \neq 0\}$.

      i. Prove that $G$ is a group under the usual matrix multiplication.

      ii. Show that $G$ is **not an abelian group.**

   (c) Let $x$ be an element of the group $G$. Prove that if $|x| = n$ for some positive integer $n$, then $x^{-1} = x^{n-1}$.

2. (a) Let $H$ be a finite nonempty subset of a group $G$ such that $HH \subset H$. Prove that $H \leq G$. ($HH = \{h_1 * h_2 \mid h_1, h_2 \in H\}$, where $*$ denotes the binary operation defined on $G$.)

   (b) i. State **Lagrange's theorem**.

      ii. Prove that if $G$ is a finite group and $x \in G$, then $x^{|G|} = 1$ for all $x \in G$. (1 denotes the identity element of the group $G$.)

   (c) Prove that if $H$ and $K$ are normal subgroups of a group $G$ then their intersection $H \cap K$ is also a normal subgroup of $G$.

3. (a) Define $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $\psi((x, y)) = x$. Prove that $\psi$ is a homomorphism. Find the Kernal of $\psi$.

   (b) Let $G$ and $H$ be groups and let $\phi : G \longrightarrow H$ be a homomorphism. Prove that, in the usual notation $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.

   (c) i. Let $G$ and $H$ be finite groups. If $\phi : G \longrightarrow H$ is an isomorphism, prove that $|G| = |H|$.

      ii. Determine whether $\mathbb{Z}_3$ and $S_3$ are isomorphic in the usual notation.

Continued.

4. (a) Let $\mathbb{Q}(\sqrt{2}) = \{a + \sqrt{2}b \mid a, b \in \mathbb{Q}\}$ be a subset of $\mathbb{R}$.
   i. Prove that $\mathbb{Q}(\sqrt{2})$ is a commutative ring, with identity 1, under the usual addition and multiplication of real numbers.
   ii. Show that every nonzero element in this commutative ring is a unit.

   (b) Let $R$ be a ring. Prove that in the usual notation that $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$.

   (c) Let $R$ be a ring with identity 1. Prove that if $u$ is a unit in $R$ then $-u$ is also a unit in $R$.

5. (a) Let $R$ be a ring with identity 1. For a fixed element $a \in R$ define $C(a) = \{r \in R \mid ra = ar\}$. Prove that $C(a)$ is a subring of $R$ containing $a$.

   (b) Let $R$ be a ring and $a \in R$ be such that $a^2 = a$. Define $\phi_a : \mathbb{Z} \longrightarrow R$ by $\phi_a(x) = xa$. Prove that $\phi$ is a ring homomorphism.

   (c) Determine the **characteristics** of the the rings, $\mathbb{Q}$ and $\mathbb{Z}_8$.