University of Kelaniya - Sri Lanka
Center for Distance & Continuing Education
Bachelor of Science(General) Degree
Third Examination - 2022 (2024 September) First Semester (External)
(New Syllabus)
Faculty of Science

Pure Mathematics
PMAT 36593 - Complex Variables

No.of Questions: Six(06) No.of Pages: Three(03) Time: $(2\frac{1}{2})$hrs
Answer Five(05) Questions Only

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1. (a) Express the following complex numbers in the form $a + ib$ where $a, b \in \mathbb{R}$.

   (A). $\dfrac{i}{1-i} + \dfrac{1-i}{i}$ (B). $\dfrac{(1+i)^{2024}}{\sqrt{2}}$

   (b) Compute the value of

   $$\arg\left(\frac{i}{-2-2i}\right).$$

   (c) Find the polar form of $z = -\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$, and write it as $z = re^{i\theta}$.

   (d) Find all the complex roots of $z = (-1+i)^{1/3}$ and locate them graphically.

2. (a) Show that $\lim\limits_{z \to 0} \dfrac{z}{\bar{z}}$ does not exist.

   (b) Determine whether the complex-valued function $f(z)$ is continuous at $z = 0$, where

   $$f(z) = \begin{cases} \dfrac{z^2}{|z|^2} & z \neq 0 \\ 1 & z = 0. \end{cases}$$

   (c) Show that, if the function $f(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0.

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3. (a) Let $f(z) = u(x,y) + iv(x,y)$ be defined on a domain D in the complex plane, where $u(x,y)$ and $v(x,y)$ are real-valued functions. Assume that first-order partial derivatives of u and v exist, continuous, and satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   i. Show that $f = u + iv$ is analytic.
   ii. Determine whether the function $f(z) = \mathrm{Re}(z) = x$ is analytic.

   (b) i. Show that $u(x,y) = \ln(x^2 - y^2)$ is harmonic on $\mathbb{C}$.
   ii. Find the harmonic conjugate $v(x,y)$ for $u(x,y)$ on $\mathbb{C}$ where $v(0,0) = 0$. Express $f(z) = u + iv$ in terms of z.

4. (a) Calculate

$$\int_\gamma \bar{z}dz,$$

   where γ denotes the circle$|z - i| = 2$ oriented counterclockwise.

   (b)

   i. State Cauchy's Theorem.
   ii. Evaluate $\oint_\gamma \frac{dz}{(z-a)}$, where γ is any simple closed curve and $z = a$ is

   (A). outside γ (B). inside γ.

   (c) Using Cauchy's integral formula, prove that

$$\oint_\gamma \frac{z}{z^2+9}dz = \pi i\ ,$$

   where γ is the circle $|z - 2i| = 4$.

5. (a) Find the Laurent series of $f(z) = \dfrac{z^2}{z^2 - 3z + 2}$ in each of the following domains

   i. $1 < |z| < 2$,
   ii. $1 < |z - 3| < 2$.

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(b) Consider the complex function $f(z) = \dfrac{e^z}{1 - z^2}$

i. Find all its singularities in $\mathbb{C}$.

ii. For each singularity, determine whether it is a pole, a removable singularity, or an essential singularity.

iii. Compute the residue of the function at each singularity.

iv. Write the principal part of the function at each singularity.

6. (a) State the Residue theorem.

(b) Using the Residue theorem, evaluate the integral

$$\oint_\gamma \frac{2z+6}{z^2+4}\, dz,$$

where γ is the circle $|z - i| = 2$.

(c) Show that,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi.$$

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