UNIVERSITY OF KELANIYA - SRI LANKA

CENTER FOR DISTANCE & CONTINUING EDUCATION

BACHELOR OF SCIENCE (GENERAL) EXTERNAL

Third year Second semester examination - 2019 (March 2025)

(New Syllabus)

Faculty of Science

Applied Mathematics

AMAT 37613 - Mathematical Modeling

No. of Questions: **Six (06)** No. of Pages: **Three (03)** Time: Two & half (**2.5**) hrs

Answer **Five (05)** Questions only.

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1. Suppose that $P(t)$ is the number of individuals in a population of bacteria having constant birth and death rates β and δ (in births or deaths per individual per unit of time).

   (a) Show that the mathematical model for the bacteria growth is

   $$\frac{dP}{dt} = KP$$

   where $K = \beta - \delta$.

   (b) Solve the differential equation for $P(t)$ given the initial condition $P(0) = P_0$.

   (c) According to data listed at www.census.gov, the world's total population reached 6 billion persons in mid-1999, and was then increasing at the rate of about 212 thousand persons each day. Assume that natural population growth at this rate continues.

      i. What is the annual growth rate K?

      ii. What will be the world population in 2050?

   (d) Radioactive decay is the process by which an unstable atomic nucleus loses energy by radiation. The half-life τ of a radioactive isotope is the time required for half of it to decay.

      i. Develop a mathematical model for the radioactive decay and define all variables and parameters used in your model.

      ii. Obtain a formula for half-life τ.

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2. (a) Explain the significance of the logistic growth model in population dynamics. How does it differ from exponential growth model?

   (b) Consider the logistic differential equation for a fish population $P(t)$

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - h$$

   where r is the intrinsic growth rate, K is the carrying capacity, and h is the constant harvesting rate.

      i. Find all the equilibria.
      ii. Determine the stability of these equilibria.
      iii. Find the maximum sustainable harvest rate h_{max}.

   (c) A lake has a carrying capacity of 5000 fish and an intrinsic growth rate of 0.2 per year. If a constant harvesting rate of 600 fish per year is introduced, determine the population survives in the long run.

3. (a) A lake receives inflow from an upstream source while simultaneously discharging into a downstream body. Suppose a pollutant with initial concentration C_i grams of pollutant per liter is introduced into the lake, and its concentration changes over time. Volume of the lake is $V(t)$ at time t. Pollutant flows into the lake at the constant rate of r_i liters per second, and that the pollutant in the lake thoroughly mixed by stirring and flows out at the constant rate of r_o liters per second. And the amount of pollutant that flows out of the lake during the same time interval depends on the concentration C_o at time t.

      i. Show that the pollutant $x(t)$ in the lake satisfies the differential equation

$$\frac{dx}{dt} = r_i C_i - \frac{r_o}{V} x.$$

      ii. Assume that Lake Erie has a volume of $480km^3$ and that its rate of inflow(from Lake Huron) and out flow(to Lake Ontario) are both $350km^3$ per year. Suppose that at the time $t = 0$(years), the pollutant concentration of Lake Erie is five times that of Lake Huron. If the out flow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

4. The Lotka-Volterra competition system is given by the following equations:

$$\frac{dx_1}{dt} = x_1(1 - x_1 - \alpha x_2)$$

$$\frac{dx_2}{dt} = x_2(1 - \alpha x_1)$$

where $\alpha > 0$ and x_1 and x_2 are the population of two species X_1 and X_2, respectively.

   (a) Find all the equilibria.

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(b) Determine the stability of boundary equilibria using Jacobian matrix.

(c) Draw the nullclines and graph the direction of flow along each of the nullclines.

5. (a) Consider the Ricker model

$$N_{t+1} = N_t \exp\left[r\left(1 - \frac{N_t}{K}\right)\right]$$

where the parameter r is the intrinsic growth rate and K is the carrying capacity.

i. Determine all equilibria.

ii. Determine the stability of the equilibria.

(b) Let H_t denote the host and P_t the parasite populations at time t. Then the host-parasite model is defined as follows:

$$H_{t+1} = \frac{\alpha_1 H_t}{1 + \gamma_1 P_t}$$

$$P_{t+1} = \frac{\alpha_2 P_t}{1 + (\gamma_2 P_t / H_t)}$$

where $H_0 > 0, P_0 > 0, \alpha_i > 0$, and $\gamma_i > 0, i = 1, 2$. The parameters α_i are growth rates of the host and parasite populations in the absence of the other population

i. Assume $\alpha_i > 1$ for $i = 1, 2$, then find the unique positive equilibria (H^*, P^*) of the system.

ii. Perform a local stability analysis of the (H^*, P^*) using the Jacobian matrix.

6. (a) Consider the following difference equation

$$x_{t+2} - 9x_{t+1} + 20x_t = 0.$$

i. Classify the difference equation as to order, linear or nonlinear, autonomous or nonautonomous.

ii. If the equation is linear, determine if it is homogeneous or nonhomogeneous.

iii. Find the general solution to the above equation.

(b) Consider

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}.$$

i. Using the discrete Putzer algorithm, evaluate A^n.

ii. Find the solution of the difference system $x(n+1) = Ax(n)$, $x(0) = (1, 2)^T$.

— End of Examination Paper —