University of Kelaniya - Sri Lanka
Center for Distance & Continuing Education
Bachelor of Science (General) External
Second Year second semester Examination - 2019 (2024 September)
(New Syllabus)
Faculty of Science

Applied Mathematics
AMAT 36593 - Computational Mathematics

No.of Questions: Six(06) **No.of Pages: Four(04)** **Time: Two & half($2\frac{1}{2}$)hrs**

Answer Five(05) Questions Only

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1. (a) Starting with the Taylor series expansions of $u(x, t-\Delta t)$, $u(x, t-2\Delta t)$, derive the **second order left sided finite difference** approximation for $\dfrac{\partial u}{\partial t}$ as

$$\frac{3u(x,t) - 4u(x, t-\Delta t) + u(x, t-2\Delta t)}{2\Delta t}.$$

Further show that the truncation error for the above approximation is given by

$$\frac{\Delta t^2}{3}\frac{\partial^3 u}{\partial t^3} + \text{terms of higher order}$$

[35 marks]

(b) Consider the following 1D Heat equation with initial and boundary conditions:

$$u_t + \gamma u_{xx} = 0 \qquad 0 < x < 1$$
$$u(0,t) = g_0(t),\ u(1,t) = g_1(t) \qquad t > 0$$
$$u(x,0) = \eta(x).$$

(i) Derive a finite difference scheme to solve the above problem by approximating the space derivative by the **second order symmetric difference** approximation and time derivative by **second order left sided finite difference** (part (a)) approximation.

[30 marks]

(ii) Represent the discrete form of the scheme in a matrix form.

[35 marks]

Continued...

2. (a) What it meant by saying that the partial differential equation (PDE) is well posed? **[10 marks]**

(b) Define the following terms with respect to the finite difference scheme:

(i) Consistency

(ii) Convergence

(iii) Stability

[30 marks]

(c) The finite difference representation of the 1D Heat equation is given by

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mu \, \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

where μ is a positive constant and Δx, Δt are step sizes of space and time respectively.

(i) By taking $\alpha = \mu \dfrac{\Delta t}{\Delta x^2}$, show that the above scheme is consistent with the 1D Heat equation $u_t = \mu u_{xx}$.

[30 marks]

(ii) Discuss the stability of the above scheme by using von-Neumann stability analysis. If the scheme is von-Neumann stable, find the condition for α.

[30 marks]

3. Consider the following Laplace equation:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < 1, \quad & 0 < y < 1 \\ u(x,0) &= x, & u(x,1) = x + 1, \quad & 0 \le y \le 1 \\ u(0,y) &= y, & u(1,y) = y + 1 \quad & 0 \le x \le 1. \end{aligned}$$

To construct the grid use 4 subintervals for x spatial variable and 2 subintervals for y spatial variable.

Find the numerical solution of each grid point and compare those with actual solution of $u(x,y) = x + y$.

[100 marks]

Continued...

4. Consider the following initial-boundary valued problem;

$$
\begin{aligned}
u_t + \mu u_x &= u_{xx} \qquad 0 < x < l \\
u(0,t) = g_1(t),\ u_x(l,t) &= g_2(t) \qquad t > 0 \\
u(x,0) &= h(x) \qquad 0 \le x \le l.
\end{aligned}
$$

(i) Derive the finite difference scheme by using centered difference to approximate the first derivatives of t, forward difference to approximate the first derivative of x except at $x = l$, second order symmetric differences to approximate second derivative and backward difference to approximate the boundary condition at $x = l$.

[60 marks]

(ii) Represent the discrete form of the scheme in a matrix form.

[30 marks]

(iii) What conditions need to be satisfied by the boundary value problem and boundary conditions in order to apply the von-Neumann stability analysis?

[10 marks]

5. (a) Consider the differential equation $\mathcal{L}\varphi + f = 0$ in the domain D. Let $\psi(x) = \sum_{i=1}^{n} c_i N_i(x)$ be the approximation solution for the differential equation, where c_i, N_i are coefficients and trial functions, respectively. Explain the steps used to evaluate the approximate solution by using the weighted residual method.

[25 marks]

(b)

(i) In the usual notations, define the weak derivative of a function. Find the weak derivative of the following function $f(x)$ in the domain $(0, 2)$:

$$
f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & 1 \le x < 2. \end{cases}
$$

[30 marks]

(ii) In the usual notations, define the vector spaces $L^2(a,b)$ and $H_0^1(a,b)$.

[20 marks]

(iii) Find the weak form of the following boundary value problem:

$$
\begin{aligned}
u_x(x) + \epsilon u_x u_{xx}(x) &= h(x) \\
u(0) = 0,\ u(l) &= 0.
\end{aligned}
$$

Hint: $2u_x u_{xx} = \dfrac{du_x^2}{dx}$.

[25 marks]

Continued...

6. (a) Consider the following boundary value problem:

$$u'' + xu' = x^2, \quad 0 < x < 1$$
$$u(0) = 0, \ u(1) = 0.$$

(i) For the point collocation method, which function need to be selected as the weighted function?

[10 marks]

(ii) Taking the trial function as $u(x) = x(1-x)(c_0 + c_1 x)$, solve the above problem using point collocation method at the points $x = \frac{1}{3}, \frac{2}{3}$.

[40 marks]

(b) Consider the following boundary value problem:

$$u'' = \sec^2(\pi x)\csc(\pi x) + x, \quad 0 < x < 1$$
$$u(0) = 0, \ u(1) = 0.$$

(i) For the Galerkin method, which function need to be selected as the weighted function?

[10 marks]

(ii) Taking the trial function as $u(x) = A\sin(\pi x)$, solve the above problem using Galerkin method.

[40 marks]

— End of Examination Paper —