University of Kelaniya - Sri Lanka
Center for Distance & Continuing Education
Bachelor of Science(General) External
Second Year second semester Examination - 2023 (2026 July)
(New Syllabus)
Faculty of Science

Applied Mathematics
AMAT 27572 - Numerical Methods II

No.of Questions: Five(05) No.of Pages: Four(04) Time: Two(2)hrs
Answer Four(04) Questions Only

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1. (a) (i) In the usual notations state L_1 norm, L_2 norm and L_∞ norm of $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ defined on $\mathbb{R}^n$.

(ii) Briefly explain why this ordering $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2 \leq ||\mathbf{x}||_1$ holds in general.

(iii) Let $\mathbf{x} = (3, -4, 12)^T$ be a vector in $\mathbb{R}^3$, compute the norms of $||\mathbf{x}||_1$, $||\mathbf{x}||_2$ and $||\mathbf{x}||_\infty$.

(b) (i) State the four properties that a real-valued function must satisfy to be a matrix norm.

(ii) Verify that the function $\|\cdot\|_\infty$ defined on $\mathbb{C}^{m\times n}$ by

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|,$$

is a norm on $\mathbb{C}^{m\times n}$, where $A = (a_{ij}) \in \mathbb{C}^{m\times n}$.

(c) (i) Find the condition number of the following system of equations:

$$\begin{aligned} 2x + \ y &= -1 \\ x + 3y &= 2\,. \end{aligned}$$

(ii) How do you determine the given system of equations is ill conditioned or well conditioned?

Continued...

2. (a) Approximate $\int_0^1 \frac{1}{1+x^2}\, dx$ to the first decimal place using the Trapezoidal rule.

(b) The two-point Gauss quadrature rule approximates an integral over $[-1, 1]$ as

$$\int_{-1}^{1} g(t)\, dt \approx w_1\, g(t_1) + w_2\, g(t_2).$$

(i) By requiring this formula to be exact for all polynomials $g(t)$ of degree ≤ 3 (i.e. testing it on $g(t) = 1,\ t,\ t^2,\ t^3$), derive a system of equations relating w_1, w_2, t_1, t_2.

(ii) Solve the system of part (i) to show that

$$w_1 = w_2 = 1, \qquad t_1 = -\frac{1}{\sqrt{3}}, \qquad t_2 = \frac{1}{\sqrt{3}}.$$

(iii) Evaluate the following integral using the two-point Gauss quadrature approximation formula,

$$I = \int_0^1 \frac{1}{1+x^2}\, dx.$$

3. (a) Consider the following system of linear equations;

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -x_1 + 11x_2 - x_3 &= 25 \\ 2x_1 - x_2 + 10x_3 &= -11. \end{aligned}$$

(i) Show that the coefficient matrix of the above system is strictly diagonally dominant, and state what this guarantees about the existence, uniqueness, and convergence of iterative solutions for this system.

(ii) Using the Gauss-Seidel method, perform three iterations to approximate the solution of the system above with the initial approximation as $(0, 0, 0)$.

Continued...

(b) (i) In the usual notations write the general formula for the Successive Over Relaxation (SOR) method.

(ii) Explain the Successive Over Relaxation method and the Successive Under Relaxation method using weighted value.

(iii) Solve the following linear system using the Successive Over Relaxation (SOR) method with the initial guess as $(0, 0)$. Take the weighted value appropriately and calculate the solution until the second iteration.

$$4x + y = 7,$$
$$x + 3y = 5.$$

4. Consider the following initial value problem:

$$\frac{dy}{dx} = x + y, \qquad y(0) = 1.$$

Using a step size of $h = 0.2$, approximate the value of $y(0.2)$ using the following numerical methods. Show all intermediate calculations clearly.

(a) Use the Taylor series method of order three to obtain an approximation for $y(0.2)$.

(b) Use Heun's method to approximate the solution at $x = 0.2$.

(c) Use the classical fourth-order Runge–Kutta method (RK4) to approximate the solution at $x = 0.2$.

(d) The exact solution of the given initial value problem is

$$y(x) = 2e^x - x - 1.$$

Calculate the exact value of $y(0.2)$ and compute the absolute errors of the approximations obtained in parts (a)–(c). Determine the most accurate numerical method according to the results.

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5. (a) In the usual notations,

   (i) define Well posdness of the initial valued problem.

   (ii) define the Lipschitz condition of a function.

   (iii) state the theorem for finding error bounds for the Euler's method.

   (b) Consider the following initial valued problem with the initial condition,

$$\begin{cases} f(t,y) = t - y + 2; \quad 0 \leq t \leq 1 \\ y(0) = 1. \end{cases}$$

   (i) Show that the function f satisfies the Lipschitz condition.

   (ii) Hence, show that the given initial value problem is well-posed on $D = \{(t,y) \mid 0 \leq t \leq 1,\ -\infty < y < \infty\}$.

   (iii) Find the error bound for the Euler's method to approximate the above IVP with $h = 0.25$. Consider the exact solution as $y(t) = t + 1 + e^{-t}$.

— End of Examination Paper —