University of Kelaniya – Sri Lanka
Centre for Distance & Continuing Education
Bachelor of Science (General) External
Second year second semester examination - 2023 (2026 July)
(New Syllabus)
Faculty of Science
Pure Mathematics
PMAT 27583- Functions of Several Variables

No. of Questions: Six(**06**) No. of Pages:Three(03) Time: Two and half ($2\frac{1}{2}$)**hrs**

Answer **Five (05)** questions only.

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01. (a) State the $\varepsilon - \delta$ definition of $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

[10 Marks]

Using $\varepsilon - \delta$ definition, show that $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$.

[20 Marks]

(b) Prove that if the double limit of a function $f(x,y)$ at a point (a,b) exists, then it is unique.

[30 Marks]

Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

[10 Marks]

(c) Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{(y-x)(1+x)}{(y+x)(1+y)}, \; y + x \neq 0, -1 < x, y < 1.$$

(i) Show that $\lim_{x\to0}\lim_{y\to0} f(x,y) \neq \lim_{y\to0}\lim_{x\to0} f(x,y)$.

[20 Marks]

(ii) Determine whether the $\lim_{(x,y)\to(0,0)} f(x,y)$ exists .
Justify your result.

[10 Marks]

02. (a) Define the continuity of a function of two variables $f(x,y)$ at the point (a,b).

[05 Marks]

Let $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

Show that the function is continuous at the point (0,0).

[20 Marks]

Continued...

(b) Define the differentiability of a function $f(x,y)$ at the point (a,b).

[05 Marks]

Let $f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

Show that

(i) $f_x(0,0) = f_y(0,0) = 0$

[20 Marks]

(ii) $f(x,y)$ is not differentiable at the point $(0,0)$.

[35 Marks]

(c) Let the function $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^3 + 2x^2y^2 + y^3.$$

Find the mixed second order partial derivative $\frac{\partial^2 f}{\partial y \partial x}$ at the point $(1,2)$.

[15 Marks]

03. (a) Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 \quad , & (x,y) = (0,0) \end{cases}.$$

(i) Show that $f_x(0,y) = -y$ for all $y \neq 0$. [15 Marks]

(ii) Find $f_y(x,0)$ for all $x \neq 0$. [15 Marks]

(iii) Evaluate $f_x(0,0)$ and $f_y(0,0)$. [20 Marks]

(iv) Find the values of $f_{xy}(0,0)$ and $f_{yx}(0,0)$. [20 Marks]

(b) Suppose that $u = x^2y + y^3$, where $x = r\cos\theta$ and $y = r\sin\theta$.

Find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$.

Hence , show that $\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = 4x^2y^2 + (x^2+3y^2)^2$.

[30 Marks]

04. (a) State clearly a set of sufficient conditions for the function $f(x,y)$ to have a local maximum or a local minimum at the point (a,b) .

[10 Marks]

Find the stationary points of the function

$$f(x,y) = x^3 + y^2 - 3x - 6y - 1 .$$

Hence show that $f(x,y)$ has a point of minimum at $(1,3)$.

[25 Marks]

(b) By using the method of Lagrange multipliers, find the stationary points of $f(x,y) = xy$ subject to the constraint $4x^2 + y^2 = 4$.

[50 Marks]

(c) Use differentials to find an approximate value for $\sqrt{9.99 - (1.01)^3}$.

[15 Marks]

Continued...

05. (a) Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\, dxdy$.

[25 Marks]

(b) Find the value of $\iint_A (x^2 + y^2)\, dxdy$, where A is the region bounded by $x = 0, y = 0$ and $x + y = 1$.

[25 Marks]

(c) Transform the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dydx$ into polar coordinates and hence find the value of the integral.

[25 Marks]

(d) Show that $\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=1}^{2} x^2yz\, dxdydz = 1$.

[25 Marks]

06. (a) Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x + y) dxdy$ by changing the order of integration.

[40 Marks]

(b) By using a suitable transformation, evaluate $\iint_D \frac{(x+2y)}{\cos(x+y)}\, dxdy$., where D is the parallelogram bounded by the lines $y = x$, $y = x - 1$, $x + 2y = 0$ and $x + 2y = 2$.

[60 Marks]

*************THE END***************