University of Kelaniya - Sri Lanka
Center for Distance & Continuing Education
Bachelor of Science(General) External
Second year first semester examination - 2023 (2026 January)
(New Syllabus)
Faculty of Science

Pure Mathematics
PMAT 26553 - Linear algebra

No.of Questions: Six(06) **No.of Pages: Five(05)** **Time: Two $\left(2\frac{1}{2}\right)$hrs**

Answer FIVE(05) Questions only.
Question No. 01 is compulsory and underline the correct option. Four (04) other questions should be attempted from the rest of the five questions.

---

1. (a) Which set of conditions defines a basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for a vector space V?
   - i. S spans V and S is linearly independent.
   - ii. S spans V and S is orthogonal.
   - iii. S is linearly dependent and $n = \dim(V)$.
   - iv. S spans V and $\mathbf{0}$ is in S.
   - v. S is linearly independent.

   (b) If W is a nonempty subset of a vector space V, W is a subspace of V if and only if which closure conditions hold?
   - i. W is closed under scalar multiplication, and W contains the additive inverse for every vector in W.
   - ii. W contains the zero vector and W is closed under vector addition.
   - iii. If u and v are vectors in W, then $u + v$ is in W, and if u is in W and c is any scalar, then cu is in W.
   - iv. W is closed under vector addition in W.
   - v. None of the above.

   (c) Let A be an $m \times n$ matrix. The set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is called the nullspace of A, denoted by $N(A)$. The dimension of $N(A)$ is called the:
   - i. Rank of A.
   - ii. Characteristic polynomial of A.
   - iii. Nullity of A.
   - iv. Kernal of A.
   - v. None of the above.

Continued...

(d) If A is an $m \times n$ matrix of rank r, and the number of columns is n, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is given by the Rank-Nullity Theorem as:

i. $r + n$.

ii. $n - r$.

iii. $n + \text{nullity}(A)$.

iv. $m - r$.

v. $r - n$.

(e) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Which subspace is equal to the range of T?

i. The kernel of T.

ii. Eigenspace of A.

iii. The row space of A.

iv. The nullspace of A.

v. The column space of A.

(f) Let V be an inner product space. The norm (or length) of a vector u in V is defined as:

i. $\|u\| = \langle u, u \rangle$.

ii. $\|u\| = \dim(V)$.

iii. $\|u\| = |k|\|u\|$ where k is a scalar.

iv. $\|u\| = d(\mathbf{0}, u)$.

v. $\|u\| = \sqrt{\langle u, u \rangle}$.

(g) Let V be an n-dimensional inner product space. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of n nonzero vectors in V, what is necessarily true about S?

i. S is a basis for V.

ii. S must be an orthonormal set.

iii. S spans a proper subspace of V.

iv. S is linearly dependent.

v. S contains the zero vector.

(h) Let A be an $m \times n$ matrix, and let B be a row-echelon form of A. The dimension of the row space of A (which is defined as the rank of A) is equal to:

i. The number of columns in A.

ii. The dimension of the nullspace of A.

iii. The number of nonzero row vectors in B.

iv. The number of vectors used to span the column space of B.

v. $n - \text{rank}(A)$.

Continued...

(i) Let V be a vector space, and let $\langle \mathbf{u}, \mathbf{v} \rangle$ be a function that associates a real number with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$. If $\langle \mathbf{v}, \mathbf{v} \rangle = -6$ for some nonzero vector $\mathbf{v} \in V$, which inner product axiom has been violated, showing that this function is not an inner product?

i. Symmetry axiom ($\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$).
ii. Additivity axiom ($\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$).
iii. Homogeneity axiom ($c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$).
iv. Positivity axiom ($\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$).
v. Multiplicative identity axiom.

(j) Characteristic polynomial of $A = \begin{bmatrix} 1 & 5 & 7 & 8 \\ 0 & 2 & 0 & 6 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is

i. $\rho_A(x) = (x-1)(x+2)(x-3)(x-4)$
ii. $\rho_A(x) = (x-1)(x+2)(x+3)(x-4)$
iii. $\rho_A(x) = (x-1)(x+2)(x-3)(x+4)$
iv. $\rho_A(x) = (x+1)(x-2)(x-3)(x-4)$
v. $\rho_A(x) = (x-1)(x-2)(x-3)(x-4)$

Continued...

2. (a) i. Determine whether the set of all 2×2 matrices is a vector space with respect to the following operations:

$$(a, b) + (c, d) = (a + c, b + d) \text{ and } k(a, b) = (ka, b).$$

ii. Let $V = \mathbb{R}^2$, with the usual (standard) addition of vectors and the non-standard scalar multiplication defined by

$$c(x_1, x_2) = (cx_1, 0)$$

for all scalars $c \in \mathbb{R}$ and all $(x_1, x_2) \in \mathbb{R}^2$. Show that V (with these operations) is *not* a vector space over $\mathbb{R}$.

(b) Let U be a vector space and let V and W be subspaces of U.

i. Prove that the $V \cap W$ is also a subspace of U.

ii. Give one example of two subspaces whose intersection is $\{0\}$.

3. (a) Let W be a subspace of $\mathbb{R}^4$ spanned by $(1, 4, -1, 3), (2, 1, -3, -1)$, and $(0, 2, 1, -5)$

i. Find a basis of W.

ii. Find the dimension of the subspace W.

iii. Find the rank of the following matrix

$$A = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 2 & 1 & -3 & -1 \\ 0 & 2 & 1 & -5 \end{bmatrix}$$

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T(x_1, x_2) = (2x_1 - x_2,\ x_1 + 3x_2).$$

Find the matrix representation of T relative to the bases

$$B = \{(1, 1), (1, -1)\} \quad \text{and} \quad B' = \{(1, 0), (0, 1)\}.$$

4. (a) The vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (1, 1, 1)$ span a plane in $\mathbb{R}^3$. Find an orthonormal basis for this subspace.

(b) Let the vector space P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

Apply the Gram-Schmidt process to transform the standard basis $\{1, x, x^2\}$ for P_2 into an orthogonal basis $[\phi_1(x), \phi_2(x), \phi_3(x)]$.

5. (a) Let $T : P \to \mathbb{R}$ be defined by

$$T(p) = \int_{0}^{1} x\, p(x)\, dx.$$

Show that T is a linear transformation from P, the vector space of polynomial functions, into $\mathbb{R}$, the vector space of real numbers.

Continued...

(b) Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation defined by

$$T(\mathrm{x}) = A\mathrm{x}, \qquad \mathrm{x} \in \mathbb{R}^5,$$

where

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 & 1 \\ 1 & 0 & 4 & -2 & 5 \\ 0 & 2 & -3 & 1 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

i. Find a basis for $\ker(T)$ as a subspace of $\mathbb{R}^5$.
ii. Find a basis for the range of T.
iii. Justify the rank-nulity theorem for the matrix A.

6. (a) Show that the following matrix is not diagonalizable.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(b) Consider the following matrix

$$B = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$$

i. Find the eigenvalues of B.
ii. Find the eigenspaces corresponding to the eigenvalues of B.
iii. Find an invertible matrix P such that $P^{-1}BP$ is a diagonal matirx.
iv. Hence Calculate B^{2025}.

— End of Examination Paper —