University of Kelaniya – Sri Lanka
Centre for Distance & Continuing Education
Bachelor of Science (General) External
Second year First semester examination - 2019 (2025 May)
(New Syllabus)
Faculty of Science

PURE MATHEMATICS
PMAT 26553-Linear Algebra (Repeat)

No. of Questions: Six(06) No. of Pages: Three(03) Time: **Two & half (2 1/2) Hours.**

Answer **Five (05)** questions only.

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01. (a) Let V be a vector space, $\underline{u} \in V$ and $k \in \mathbb{R}$. Show that
(i) $0\underline{u} = \underline{0}$ (ii) $(-1)\underline{u} = -\underline{u}$

(b) Define a subspace W of a vector space V.

Let U and W are subspaces of a vector space V. Prove that $U \cap W$ is also a subspace of V.

(c) Let V be the vector space of all 2×2 matrices over the real field $\mathbb{R}$. Show that W is not a subspace of V where $W = \{A |\ A^2 = A\}$.

02. (a) Write the polynomial $\underline{v} = t^2 + 4t - 3$ in $P_2(\mathbb{R})$ as a linear combination of the Polynomials $\underline{e_1} = t^2 - 2t + 5$, $\underline{e_2} = 2t^2 - 3t$ and $\underline{e_3} = t + 3$.

(b) Determine whether the following vectors in $\mathbb{R}^3$ are linearly independent: $(1,2,-3), (1,-3,2), (2,-1,5)$.

(c) Define what you meant by saying that the set of vectors $S = \{\underline{v_1}, \underline{v_2}, \dots, v_n\}$ is a basis of a vector space V.

Let W be the subspace of $\mathbb{R}^5$ spanned by $\underline{u_1} = (1,2,-1,3,4)$, $\underline{u_2} = (2,4,-2,6,8)$, $\underline{u_3} = (1,3,2,2,6)$, $\underline{u_4} = (1,4,5,1,8)$ and $\underline{u_5} = (2,7,3,3,9)$.

Find a subset of the vectors that form a basis of W.

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03. (a) Let $B = \{(1,0,1), (1,1,0), (0,1,1)\}$ be a basis for $\mathbb{R}^3$. The coordinate vector of $\underline{x}$ relative to the basis B, $[\underline{x}]_B$ is $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Find the vector $\underline{x}$.

(b) Consider the two bases B and $B'$of $\mathbb{R}^3$ given by
$B = \{(1,0,2), (0,1,3), (1,1,1)\}$ and $B' = \{(2,1,1), (1,0,0), (0,2,1)\}$.

(i) Find the transition matrix from B to B' .

(ii) In the usual notation, if $[\underline{x}]_{B'} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, find $[\underline{x}]_B$.

04. (a) For $\underline{u} = (x_1, y_1), \underline{v} = (x_2, y_2) \in \mathbb{R}^2$, the product $\langle \underline{u}, \underline{v} \rangle$ is defined by
$$\langle \underline{u}, \underline{v} \rangle = x_1x_2 + 2y_1y_2.$$

Show that $\langle \underline{u}, \underline{v} \rangle$ is an inner product on $\mathbb{R}^2$.

(b) Show that $f(x) = x$ and $g(x) = \frac{1}{2}(3x^2 - 1)$ are orthogonal in the inner product space $C[-1,1]$ with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$.

(c) Apply the Gram-Schmidt orthonormalization process to find an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by $\underline{v}_1 = (1,1,1,1)$, $\underline{v}_2 = (0,1,1,1)$ and $\underline{v}_3 = (0,0,1,1)$.

05. (a) Determine whether the function $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$T(x, y, z) = (x + y, x - y, z)$ is a linear transformation.

(b) Let $B = \{(1,0,2), (0,1,3), (1,1,1)\}$ be a basis for $\mathbb{R}^3$ and $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T(1,0,2) = (-1,2)$, $T(0,1,3) = (0,4)$ and $T(1,1,1) = (1,0)$. Find $T(1,2,-1)$.

(b) Consider the linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ given by
$$T(x, y, z, t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3t).$$

(i) Find the standard matrix representation of T.

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(ii) Find the bases for $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$ of the linear transformation T and verify the Rank-Nullity theorem.

(iii) Is T one to one? Justify your answer.

06. (a) State the Cayley-Hamilton theorem for any square matrix A.

Verify the Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$.

Hence find A^{-1}.

(b) Let $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$.

(i) Find all eigenvalues of A.

(ii) Find a maximum set of linearly independent eigenvectors of A.

(iii) Is A diagonalizable? If yes, find P where $P^{-1}AP$ is diagonal.

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