UNIVERSITY OF KELANIYA - SRI LANKA

FACULTY OF SCIENCE

Bachelor of Science (General) Degree Examination (External) - 2017

Academic Year 2013/2014

PMAT 102 – Pure Mathematics

No. of Questions: Eight (08) No. of Pages: Three (03) Time: Three (03) hrs.

Answer six (06) Questions only.

---

1. (a) Let $S = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$. Find $\inf S$ and $\sup S$.

   (b) If $S \subseteq T \subseteq \mathbb{R}$, where $S \neq \emptyset$, then show that

   (i) if T is bounded above then $\sup S \leq \sup T$

   (ii) if T is bounded below then $\inf T \leq \inf S$.

   (c) State the completeness property of the real numbers.

   Let $a, b \in \mathbb{R}$. Show that if $a \leq b + \frac{1}{n}$, for all $n \in \mathbb{N}$, then $a \leq b$.

2. (a) By using the $\varepsilon - N$ definition of a limit of a sequence, show that

   $$\lim_{n\to\infty} \left(\frac{2n^2-5}{3n^2+7n}\right) = \frac{2}{3}.$$

   (b) Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be real sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \ell$, where ℓ is finite. Prove that $\lim_{n\to\infty} b_n = \ell$.

   (c) Evaluate the limits of the following sequences (s_n):

   (i) $s_n = \frac{1}{n \log n}$ (ii) $s_n = \left(1 + \frac{3}{n}\right)^n$ (iii) $s_n = \frac{3+2\sqrt{n}}{\sqrt{n}}$

Continued...

3. (a) Show that the monotonically decreasing bounded below sequences are convergent.

(b) A sequence s_n is defined by the recursive formula $s_{n+1} = \frac{1}{4-s_n}$ and $s_1 = 3$.

Show that

(i) $1 \leq s_n \leq 3$ for all $n \in \mathbb{N}$
(ii) (s_n) is a monotonically decreasing
(iii) (s_n) is convergent.

Find $\lim_{n\to\infty} s_n$.

4. A function f is twice differentiable on an open interval containing the point c.

If $f''(c) < 0$, then show that the graph of f is concave downward at the point $P(c, f(c))$.

Suppose that a function f is defined by $f(x) = \frac{x-4}{x^2}$.

(i) Use the second derivative test whenever applicable to determine the extrema of f.
(ii) Discuss the concavity of f and find the points of inflection if any.
(iii) Determine the horizontal and vertical asymptotes of f if any.
(iv) Sketch the graph of f.

5. (a) Consider the geometric series $\sum_{n=0}^{\infty} x^n$.

(i) Show that the series is convergent if $|x| < 1$ and that it is divergent if $|x| \geq 1$.

(ii) Show that the sum of the series is $\frac{1}{1-x}$ in case $|x| < 1$.

(iv) Find the value of α that satisfies the following equation:
$$\sum_{n=2}^{\infty}(1+\alpha)^{-n} = 2.$$

(b) Determine whether each of the following series is convergent or divergent:

(i) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$ (ii) $\sum_{n=1}^{\infty} ne^{-n^2}$.

Continued...

6. a) Test, giving sufficient reasons, whether each of the following series is convergent:

(i) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{1/n}}{n}$ (ii)` $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{2n}}$.

b) Define the absolute convergence and conditional convergence of a series.

Determine, giving sufficient reasons, whether each of the following series is absolutely convergent, conditionally convergent or divergent:

(i) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\ln n}$ (ii) $\sum_{n=1}^{\infty}(-1)^n e^{-e^2}$ (iii) $\sum_{n=1}^{\infty}\frac{(-1)^n \tan^{-1} n}{n^2+1}$

7. (a) Show that e^x and xe^x are solutions to $y'' - 2y' + y = 0$.

Hence solve $y'' - 2y' + y = \frac{e^{2x}}{(1+e^x)^2}$.

(b) Show that $x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$ is exact and find its general solution

(c) By using substitution $v = \frac{y}{x}$ or any other method, find the general solution to the differential equation: $(x+y)\frac{dy}{dx} = x - y$.

8. (a) Use Laplace transformation to solve the following differential equations

(i) $\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = 0$

(ii) $t\,\frac{d^2y}{dt^2} + (t-1)\frac{dy}{dt} - y = 0$

(b) Solve for $t \geq 0$ the following simultaneous first-order differential equations

$$\frac{dx}{dt} + \frac{dy}{dt} + 5x + 3y = e^{-t}$$

$$2\,\frac{dx}{dt} + \frac{dy}{dt} + x + y = 3$$

Subject to the initial conditions: $x = 2$ and $y = 1$ at $t = 0$.

__________//__________