University of Kelaniya – Sri Lanka

Centre for Distance and Continuing Education

Bachelor of Science (General) Degree, Third Examination (External) - 2012/2013 (New Syllabus)

June/July - 2017

Pure Mathematics – PMAT E1035

Discrete Mathematics I

No. of Questions: **Eight** (08) Number of Pages: **Four** (04) Time: **Three** (03) **Hrs**

Answer **Six** (06) Questions Only.

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1. Let A and B be two non-empty bounded subsets of $\mathbb{R}$.

   (a) Define $\sup A$ and $\inf A$.

   (b) Find $\sup A$ and $\inf A$ considering each of the following subsets A of $\mathbb{R}$.

   (i) $A = \{x : \sqrt{2} \leq x < 2 \text{ or } x = 3\}$

   (ii) $A = \left\{\frac{1}{n+1} : n \in \mathbb{Z}^+\right\}$

   (iii) $A = \left\{\frac{n}{n+1} : n \text{ is a prime number}\right\}$

   (c) The subset C is defined by $C = A + B = \{a + b : a \in A, b \in B\}$.

   (i) Show that $\sup C = \sup A + \sup B$ and

   (ii) find $\sup C$ when $A = \left\{\frac{(-1)^n}{n} : n \in \mathbb{Z}^+\right\}$ and $B = \left\{\frac{2}{n} : n \in \mathbb{Z}^+\right\}$.

2. (a) By using the definition of a limit of a sequence, prove that

$$\lim_{n\to\infty} \frac{5n}{n+3} = 5$$

(b) The sequence $(a_n)_{n=1}^{\infty}$ is defined by $a_1 = 4$ and, for $n \geq 1$, $a_{n+1} = \frac{5}{6-a_n}$.

Using mathematical induction, show that

(i) $1 < a_n < 5$ and that

(ii) $(a_n)_{n=1}^{\infty}$ is a monotone decreasing sequence

and then prove that

(iii) the sequence $(a_n)_{n=1}^{\infty}$ converges to 1.

3. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be a given function and let $a \in \mathbb{R}$. Define what is meant by $\lim_{x\to a} f(x) = l$.

(b) By using the first principles, prove that $\lim_{x\to 0}(x^2 \sin x) = 0$.

(c) Let f, g and h be three real-valued functions whose domain of each is $\mathbb{R}$. If, for all $x \in \mathbb{R}$,

$$f(x) \leq g(x) \leq h(x) \text{ and } \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$$

then prove that $\lim_{x\to a} g(x) = l$.

(c) Determine whether the function f defined below is continuous at $x = 0$.

$$f(x) = \begin{cases} xe^{-\frac{1}{x^2}} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

4. (a) State Rolle's Theorem.

(b) State Mean Value Theorem and prove it using Rolle's Theorem.

(c) Show that the equation $e^x + x = 0$ has exactly one real root.

(d) Find each of the following limits, if exists:

(i) $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4}$ (ii) $\lim_{x \to \infty} \left(1 - \frac{5}{2x}\right)^{4x}$

5. (a) Find the arc length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = \ln 3$.

(b) R is the region bounded by the curve $y = 1 + \sin(x^2)$ and the lines $y = x$, $x = 0$ and $x = \sqrt{\frac{\pi}{2}}$.

(i) Sketch R.

(ii) Find, by using the cylindrical shell method, the volume of the solid generated by revolving R about the y-axis by 360°.

(iii) Find an expression, in terms of an integral, for the volume of the solid generated by revolving R about the x-axis by 360°.

6. (a) Evaluate each of the following definite integrals:

(i) $\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$ (ii) $\int_2^5 \frac{1}{\sqrt{x^2 - 4x + 13}}\, dx$

(b) For positive integral n, if $I_n = \int x^n \sqrt{1 - x}\, dx$ then show that

$$(2n + 3)I_n = 2nI_{n-1} - 2x^n(1 - x)^{\frac{3}{2}}$$

and hence evaluate $\int_0^a x^2\sqrt{1 - x}\, dx$.

7. (a) Solve each of the following differential equations:

(i) $x\frac{dy}{dx} = \frac{y}{\ln x}$

(ii) $\frac{dy}{dx} = \frac{y\cos\frac{y}{x} - x\sin\frac{y}{x}}{x\cos\frac{y}{x}}$

(b) Show that the differential equation $(3x^2y^2 + x^2)\,dx = (2x^3y + y^2)\,dy$ is exact and then solve it.

8. Solve each of the following differential equations:

(a) $\frac{dy}{dx} + \frac{y}{x} = x^3$

(b) $\frac{dx}{dy} + xy = x^3y$

(c) $\frac{dy}{dx} = y\tan x - y^2\sec x$

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