University of Kelaniya – Sri Lanka

Centre for Distance and Continuing Education

Bachelor of Science (General) Degree First Examination – 2012

October/November 2015

Pure Mathematics - PMAT- E 1035

<u>Answer Six (06) questions only</u>

No. of questions : 08 No. of Pages : 03 Time : 03 Hours

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1) (a) Show that the set $\{x \in \mathbb{R}: 1 < x < 2\}$ has no maximum.

(b) Determine if the following subsets of $\mathbb{R}$ are bounded. Justify your answers

(i) $\{x : |x-5| \leq 2\}$ (ii) $\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}$ (iii) $\left\{\frac{1}{x^2} : x \in \mathbb{R}\backslash\{0\}\right\}$

(c) Define what is meant by a sequence $(s_n)_{n=1}^{\infty}$ of real numbers converges to a real number s. By using the definition, prove that $\lim_{n \to +\infty} \frac{5n+1}{8n-7} = \frac{5}{8}$.

2) (a) Show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$.

(b) Evaluate $\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$.

(c) Find the limit $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{|x|}\right)$, if exists.

(d) Let the function f be defined by

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x < 2 \\ \alpha x^2 - \beta x + 3 & \text{if } 2 \leq x < 3 \\ 2x - \alpha + \beta & \text{if } x \geq 3 \end{cases}$$

Find the values α and β so that f is continuous everywhere on $\mathbb{R}$.

3) (a) If f is differentiable function at point a, then prove that f is continuous at a.

(b) Let the function f be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \frac{1}{5-x} & \text{if } x \geq 4 \end{cases}.$$

(i) Find $f'_{-}(4)$ and $f'_{+}(4)$.

(ii) Sketch the graph of f.

(iii) Where is f discontinuous?

(iv) Where is f not differentiable?

(c) Find the points on the curve $y = (\cos x)/(2 + \sin x)$ at which the tangent is horizontal.

4) (a) If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \left(\frac{dy}{du}\right)\frac{d^2u}{dx^2}.$$

(b) If $F(x) = f\left(xf\left(xf(x)\right)\right)$, where $f(1) = 2, f(2) = 3, f'(1) = 4, f'(2) = 5$ and $f'(3) = 6$, find $F'(1)$.

(c) Find the slope of the tangent line to the curve $\tan(x + y) = \sin(xy)$ at the point $(\sqrt{\pi}, \sqrt{\pi})$.

5) (a) If the function f has a local maximum at point c and if $f'(c)$ exists, then prove that $f'(c) = 0$.

(b) State Rolle's Theorem and the Mean Value Theorem for a function f defined on the interval $[a, b]$. Show that

(i) if $3 \leq f'(x) \leq 5$ for all values of x, then $18 \leq f(8) - f(2) \leq 30$, and

(ii) the equation $2x - 1 - \sin x = 0$ has exactly one real root.

6) (a) Sketch the graph of a function that satisfies all of the given conditions:

$$f'(0) = f'(2) = f'(4) = 0,$$

$$f'(x) > 0 \text{ if } x < 0 \text{ or } 2 < x < 4,$$
$$f'(x) < 0, \quad \text{if } 0 < x < 2 \text{ or } x > 4$$
$$f''(x) > 0 \text{ if } 1 < x < 3, \quad f''(x) < 0 \text{ if } x < 1 \text{ or } x > 3.$$

(b) Let $f(x) = \frac{x^2}{(x-2)^2}$ for $x \in \mathbb{R}$.

Find (i) the horizontal and vertical asympotes, (ii) intervals of increase or decrease, (iii) local maximum and minimum values, and (iv) the intervals of concavity and the inflection points of f.

Skecth the grpah of f by using the above information.

7) (a) Find the following limits:

(i) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$ (ii) $\lim_{x\to 0}\frac{e^x-1-x-x^2}{x^3}$ (iii) $\lim_{x\to 0^+}(4x+1)^{\cot x}$

(b) State the Fundamental Theorem of Calculus.

Let f be defined by

$$f(x)=\begin{cases}0 & \text{if } x<0\\ x & \text{if } 0\le x\le 1\\ 2-x & \text{if } 1<x\le 2\\ 0 & \text{if } x>2\end{cases}$$ and g be defined by $g(x)=\int_0^x f(t)dt$.

(i) Find an expression for $g(x)$ for all $x\in\mathbb{R}$.

(ii) Where is g differentiable?

8) (a) Find the area of the region bounded by the curves $y=\cos x, x=0$ and $x=\frac{\pi}{2}$.

(b) Consider the region R enclosed by the curves $y=x$ and $y=x^2$. Find the volume of the resulting solid if R is rotated about

(i) the x - axis, and (ii) the line $x=-1$.

(c) Find the volume of the solid obtained by rotating the region bounded by

$y=x-x^2$ and $y=0$ about the line $x=2$.

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