කැලණිය විශ්වවිද්‍යාලය -ශ්‍රී ලංකාව

University of Kelaniya-Sri Lanka

බාහිරවිභාග අංශය

External Examinations Branch

විද්‍යා පීඨය - Faculty of Science

විද්‍යාවේදී (සාමාන්‍ය) උපාධි ප්‍රථම පරීක්ෂණය (බාහිර) - 2008 හා 2009

2010 ඔක්තෝබර්

Bachelor of Science (General) Degree First Examination (External) 2008 & 2009

October -2010

# PURE MATHEMATICS

## PMAT E1025 – Discrete Mathematics I

**No. of Questions: Eight (08)** **No. of Pages: Five (05)** **Time Allowed: Three (03) hrs**

**Answer Six (06) Questions Only**

---

1. (a) Prove that

$$A \cup B \subseteq C \Leftrightarrow A \subseteq C \text{ and } B \subseteq C$$

using fundamental properties of sets.

(b) If $A, B, C, X, Y$ are subsets of a universal set $U$, using set algebra simplify

i. $(A \cap B \cap X) \cup (A \cap B \cap C \cap X \cap Y) \cup (A \cap X \cap \overline{A})$

ii. $(A \cap B \cap C) \cup (\overline{A} \cap B \cap C) \cup \overline{B} \cup \overline{C}$.

(c) Prove that $(A \cup B) \cap C = A \cup (B \cap C)$ is **not** always true using a counter example.

(d) Let $U = \mathbb{R}$, $A = \{x \in \mathbb{R} : x > 0\}$, $B = \{x \in \mathbb{R} : x > 1\}$ and $C = \{x \in \mathbb{R} : x < 2\}$. Find $A \setminus B$ and $B \cup C$.

(e) Let $\Lambda = \{\alpha \in \mathbb{R} : \alpha > 1\}$.
If $A_\alpha = \left\{x \in \mathbb{R} : \dfrac{-1}{\alpha} \le x \le 1 + \alpha\right\}$
for each $\alpha \in \Lambda$, find $\bigcup_{\alpha \in \Lambda} A_\alpha$ and $\bigcap_{\alpha \in \Lambda} A_\alpha$, where $\mathbb{R}$ denotes the set of real numbers.



Continued...

2. Let $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Define the relation $\equiv$ on $X$ by

$$(x, y) \equiv (z, t) \Leftrightarrow xt = yz$$

for every $(x, y), (z, t) \in X$.

(i) Show that this is an equivalence relation on $X$.

(ii) Find the equivalence classes of $(0,1)$ and $(3,3)$.

(iii) Show that if $(x, y) \equiv (x', y')$ and $(z, t) \equiv (z', t')$ then $(xt + yz, yt) \equiv (x't' + y'z', y't')$.

(iv) Show that if $(x, y) \equiv (x', y')$ and $(z, t) \equiv (z', t')$ then $(xz, yt) \equiv (x'z', y't')$.

3. (a) Let $f : X \to Y$ and $g : Y \to Z$ be two functions and let $h = g \circ f : X \to Z$.

Show that

(i) If $f$ and $g$ are one-to-one, so is $h$.

(ii) If $f$ and $g$ are onto, so is $h$.

(iii) If $f$ and $g$ are bijective, so is $h$.

(iv) Let $f$ and $g$ be bijective then $h^{-1} = f^{-1} \circ g^{-1}$.

(b) (i) Let $f$ and $g$ be functions from $\mathbb{N}$ to $\mathbb{N}$ defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x > 100, \\ 2 & \text{if} \quad x \leq 100, \end{cases} \qquad \text{and}$$

$$g(x) = x^2 + 1 \quad \text{for every } x \in \mathbb{N}.$$

Determine whether each function is one-to-one and onto.

(ii) If $f : \mathbb{R}^+ \to \mathbb{R}$, $f(z) = \int_1^z \frac{dt}{t}$, show that $f$ is a one-to-one and onto function.



Continued...

4.. (a) Decide whether the following is a tautology:

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

(b) Decide whether $p \wedge (q \wedge r)$ and $(p \vee q) \wedge (q \vee r)$ are logically equivalent.

(c) By using laws of algebra of statements, show that $(p \vee q) \wedge \neg p \equiv \neg p \wedge q$ .

(d) Symbolise the following argument using

U = set of all animals.
W(x) : x is warm blooded,
C(x) : x is cold blooded,
T(x) : x has no trouble living in a cold climate,

**All animals are either warm or cold blooded.**
**Warm blooded animals have no trouble living in cold climates.**
**Therefore, the animals that do have trouble living in cold climates are cold blooded.**

Prove that the argument is valid.

5. (a) Show that

$$\det\begin{pmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{pmatrix} = 2\det\begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} \quad \text{and} \quad \det\begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix} = 2.$$

(b) Study the following system

$$\begin{aligned} x + 2my + z &= 4m \\ 2mx + y + z &= 2 \\ x + y + 2mz &= 2m^2 \end{aligned}$$

with the real parameter $m$ .



Continued...

You must determine, with proof, for which $m$ this system has

(i) no solution

(ii) unique solution

(iii) infinitely many solutions

6. (a) Let $A, B, C$ are matrices. Prove that

(i) $(AB)C = A(BC)$

(ii) $(B+C)A = BA + CA$

(b) Find all matrices $M = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ that commute with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(c) Suppose $A$ and $B$ are upper triangular matrices. Show that

(i) The product $AB$ is upper triangular

(ii) The diagonal entries of $AB$ are $a_1b_1, a_2b_2, \ldots, a_nb_n$

(d) Let $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{bmatrix}$, and $P = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$. Verify that $P^{-1}AP = \begin{bmatrix} \frac{5}{12} & 0 \\ 0 & 1 \end{bmatrix}$ and deduce that $A^n = \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} + \frac{1}{7}\left(\frac{5}{12}\right)^n \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}$.

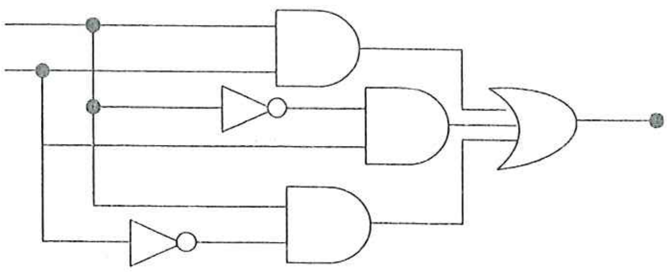
7. (a) Write down a truth table for $\overline{A}(B+C)D$.

(b) Simplify: $\overline{A}(A+B)+(B+AA)(A+\overline{B})$.



Continued...

(c)



(i) Find the Boolean expression for the above circuit.

(ii) Simplify the above Boolean expression for the circuit.

(iii) Draw a circuit for the simplified expression.

8. Fibonacci sequence is defined recursively as follows:

$$F_1 = 1, \quad F_2 = 1; \quad F_{n+2} = F_n + F_{n+1} \quad \text{for } n \geq 1.$$

(a) Suppose that $x$ is a real number such that $x^2 = x + 1$. Use the Principle of Mathematical Induction to prove that $x^n = F_n x + F_{n-1}$ for $n \geq 2$.

(b) Let $\alpha$ and $\beta$ denote the roots of the quadratic equation $x^2 = x + 1$. Show that $F_n = \frac{(\alpha^n - \beta^n)}{\alpha - \beta}$ for $n \geq 2$.

(c) Use the quadratic formula to show that the roots of the equation $x^2 = x + 1$ are given by $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

(d) Show that $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}$.

(e) Use the Principle of Mathematical Induction to show that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \quad \text{for} \quad n \geq 2.$$

-----------------------------------------//-----------------------------------------