APPLIED MATHEMATICS

Vector Algebra and Vector Analysis–AMAT E 1015

No.of questions: Eight(08)   No.of Pages: Three(03)   Time: Three(03) hrs

Answer six(06) questions only.

---

1. (a) Let $OACB$ is a parallelogram so that $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$. Show that the position vector of the point E which divides the line AB in the ratio $k{:}\,1$ is given by

$\overrightarrow{OE} = \underline{a} + \frac{k}{k+1}(\underline{b} - \underline{a})$.

Also show that extended OE cuts AC in the ratio $k{:}\,1-k$.

(b) Prove that any vector $\underline{a}$ can be written in the form

$\underline{a} = (\underline{a}\cdot\underline{i})\underline{i} + (\underline{a}\cdot\underline{j})\underline{j} + (\underline{a}\cdot\underline{k})\underline{k}$.

Hence show that $\underline{i}\times(\underline{a}\times\underline{i}) + \underline{j}\times(\underline{a}\times\underline{j}) + \underline{k}\times(\underline{a}\times\underline{k}) = 2\underline{a}$.

(c) Compute the scalar triple product $(\underline{i} - 2\underline{j} + 3\underline{k})\cdot(2\underline{i} + \underline{j} - \underline{k})\times(\underline{j} + \underline{k})$.

Are these three vectors coplanar? Justify your answer.

2. (a) Show that the vector equation of the straight line through the point A in the direction of the vector $\underline{b}$ can be written in the form $\underline{r} = \underline{a} + \lambda\underline{b}$, where λ is a parameter and $\underline{a}$ is the position vector of A with respect to the origin O.

Let the vector equation of two straight lines l_1 and l_2 be given by the equations

$\underline{r} = 2\underline{a} + \lambda(\underline{b} - 3\underline{a})$ and $\underline{r} = 3(\underline{a} - \underline{b}) - \mu(\underline{a} + \underline{b})$,

where $\underline{a}$ and $\underline{b}$ are non-collinear vectors.

(i) Find the position vector of P which is the point of intersection of lines l_1 and l_2.

(ii) Q is the point on l_1 when $\lambda = 1$ and R is the point on l_2 when $\mu = 1$. If $PQRS$ is a parallelogram , then find the vector equations of the lines QS and RS and also the position vector of the point S.

(b) Obtain the vector equation of the line joining the points $(1,-2,1)$ and $(0,3,-2)$.

Continued….

3. (a) Find the equation of the plane passing through the points $(2,2,2)$, $(3,1,1)$ and $(6,-4,6)$. Also find the perpendicular distance from the point $(2,-1,1)$ to the plane.

(b) Show that the equation of the plane passing through the origin and parallel to the vectors $\underline{i} + 2\underline{j} + 3\underline{k}$ and $2\underline{i} - \underline{j} - \underline{k}$ is $\underline{r} = (t + 2s)\underline{i} + (2t - s)\underline{j} + (3t - s)\underline{k}$, where s and t are scalar parameters.

(c) Find the vector equation of the plane passing through the point $(2,3,-1)$ and perpendicular to the vector $3\underline{i} - 4\underline{j} + 7\underline{k}$. Find the length of the perpendicular from the origin to the plane.

4. (a) A particle moves along the curve $= t^3 + 1$, $y = t^2$, $z = 2t + 5$, where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction $\underline{i} + \underline{j} + 3\underline{k}$.

(b) Let the position vector of a particle be $\underline{r} = \underline{a}\cos\omega t + \underline{b}\sin\omega t$. Show that

(i) $\underline{r} \times \frac{d\underline{r}}{dt} = \omega(\underline{a} \times \underline{b})$

(ii) $\frac{d^2\underline{r}}{dt^2} = -\omega^2\underline{r}$,

where ω is a constant and $\underline{a}$ and $\underline{b}$ are constant vectors.

(c) The acceleration of a particle at any time t is given by $e^t\underline{i} + e^{2t}\underline{j} + \underline{k}$.
Show that the velocity of the particle at $t = 0$ is $\underline{i} + \underline{j}$.

5. (a) Write down Serret-Frenet formulae.

Prove that Serret-Frenet formulae can be written in the form

$$\frac{d\underline{t}}{ds} = \underline{\omega} \times \underline{t}$$
$$\frac{d\underline{n}}{ds} = \underline{\omega} \times \underline{n}$$
$$\frac{d\underline{b}}{ds} = \underline{\omega} \times \underline{b}$$

in the usual notation . Here $\underline{\omega}$ is a vector to be determined.

(b) Find the curvature and the torsion of the curve $x = 2\log t$, $y = 4t$, $z = 2t^2 + 1$.

Continued....

6. (a) Prove that

(i) $grad(\underline{a} \cdot \underline{r}) = \underline{a}$

(ii) $grad[\underline{r}\ \underline{a}\ \underline{b}] = \underline{a} \times \underline{b}$, in the usual notation, where $\underline{a}$ and $\underline{b}$ are constant vectors.

(b) Find $div\, \underline{F}$, where $\underline{F} = (x + 3y)\underline{i} + (y - 3z)\underline{j} + (x - 2z)\underline{k}$.
Hence determine whether $\underline{F}$ is solenoidal.

(c) Show that the vector function $\underline{F} = (\sin y + z)\underline{i} + (x\cos y - z)\underline{j} + (x - y)\underline{k}$ is irrotational. Hence find a scalar function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

7. (a) Prove, in the usual notation, that

(i) $grad(r^n) = nr^{n-2}\underline{r}$

(ii) $div(\phi\underline{A}) = \phi\, div\, \underline{A} + \underline{\nabla}\phi \cdot \underline{A}$

(iii) $curl(\phi\underline{A}) = \phi\, curl\, \underline{A} + \underline{\nabla}\phi \times \underline{A}$.

(b) If $\underline{r}$ and r have their usual meanings , show that

(i) $div(r^n\underline{r}) = (n + 3)r^n$

(ii) $curl(r^n\underline{r}) = \underline{0}$

(iii) $div\left(\frac{\underline{r}}{r^3}\right) = 0$.

8. (a) State the divergence theorem.

Show that $\frac{1}{3}\int_S \underline{r} \cdot d\underline{S} = V$, where V is the volume enclosed by the surface S.

(b) Using the divergence theorem, evaluate $\int_S \underline{F} \cdot d\underline{S}$, where $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$. Here S is the surface of the cube $x = 0, x = 1$, $y = 0, y = 1, z = 0, z = 1$.

(c) Verify the Stoke's theorem for the function $\underline{F} = x\left(x\underline{i} + y\underline{j}\right)$ integrated around the square in the plane $z = 0$ whose sides are along the lines $x = 0, x = a$, $y = 0, y = a$.

****