University of Kelaniya- Sri Lanka
Faculty of Science
Centre for Distance & Continuing Education
Bachelor of Science (General) Degree
First Examination-External
March 2026

PURE MATHEMATICS | **PMAT 17543 - Theory of Calculus**
No.of Questions: Six(06) **No.of Pages:** Five(05) **Time:** Two and half ($02\frac{1}{2}$) hours

Answer only FIVE (05) questions.

1.

(a) Let S be a nonempty bounded subset of $\mathbb{R}$.

i) Prove that $\inf S \leq \sup S$.

[20 Marks]

ii) Define $-S := \{-s : s \in S\}$. Prove that $\inf S = -\sup(-S)$.

[20 Marks]

(b)

i) State the **Completeness Axiom**.

[05 Marks]

ii) If $a > 0$ and $b > 0$, prove that there exists a positive integer n such that $na > b$.

[30 Marks]

(c) For the set given below, determine whether it is bounded above, bounded below, or both. If it is bounded above (below), then find the supremum (infimum). Justify all your answers.

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

[25 Marks]
[Total 100 Marks]

Continued...

2.

(a) i. The population of a certain species in a limited environment, with initial population 100 and carrying capacity 1000, is modeled by

$$P(t) = \frac{100,000}{100 + 900e^{-t}},$$

where t is measured in years.
Find the time required for the population to reach 900.

[10 Marks]

ii. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants a, b, c, d in order that $(f \circ g)(x) = (g \circ f)(x)$ for every value of x?

[20 Marks]

(b) i. Evaluate $\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1}$.

[20 Marks]

ii. Find the values of a and b for which $\lim_{x \to 0} \frac{\sqrt{ax + b} - 2}{x} = 1$.

[25 Marks]

(c) i. State the **Squeeze Theorem**.

[05 Marks]

ii. Evaluate $\lim_{x \to \infty} e^{-2x} \cos(x)$.

[20 Marks]
[Total 100 Marks]

3.

(a) Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$.

i Find $(f \circ g)(x)$.

[05 Marks]

ii Determine whether $f \circ g$ is continuous everywhere.
Justify your answer.

[15 Marks]

Continued...

(b) For what values of b is

$$g(x) = \begin{cases} \dfrac{x-b}{b+1}, & x < 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every x?

[35 Marks]

(c)

i State the **Intermediate Value Theorem.**

[10 Marks]

ii Show that the equation $e^x + x = 3 - 2x$ has a solution in the interval $(0, 1)$.

[35 Marks]

[Total 100 Marks]

4. (a) i. Using the limit of the difference quotient, define the derivative of a function at a point.

[05 Marks]

ii. Prove that $\dfrac{d}{dx}(\sin x) = \cos x$ using the definition of the derivative.

[20 Marks]

(b) Let $u(x)$ and $v(x)$ be functions that are differentiable at x, and suppose that $v(x) \neq 0$. Show that the derivative of the quotient function $f(x) = \frac{u(x)}{v(x)}$ is

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2},$$

where $u'(x)$ and $v'(x)$ denote the derivatives of $u(x)$ and $v(x)$ with respect to x, respectively.

[20 Marks]

(c) Determine the values of a and b for which the following function differentiable for all x:

$$f(x) = \begin{cases} ax^2 + b, & x \leq 1, \\ 3x + 2, & x > 1. \end{cases}$$

[30 Marks]

Continued...

(d) Show that the following equation

$$x^3 + 2x - 2 = 0$$

has exactly one real solution.

[25 Marks]
[Total 100 Marks]

5. (a) Find equations of the tangent and normal lines to the cissoid of Diocles

$$y^2(2 - x) = x^3,$$

at the point (1, 1).

[25 Marks]

(b) Using L'Hôpital's rule, evaluate the following limits.

i. $\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$ ii. $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

[30 Marks]

(c) Let $f(x) = 2x - 1$ be defined on the interval $[a, b]$, where $a < b$.

i. Divide the interval $[a, b]$ into n equal subintervals. Write down the partition points and the width of each subinterval.

ii. Determine the lower sum L_n and the upper sum U_n of f on $[a, b]$ corresponding to the partition in part i.

iii. Evaluate the limits

$$\lim_{n \to \infty} L_n \quad \text{and} \quad \lim_{n \to \infty} U_n.$$

iv. Hence, show that $f(x) = 2x - 1$ is Riemann integrable on $[a, b]$, and compute

$$\int_a^b (2x - 1)\, dx.$$

[45 Marks]
[Total 100 Marks]

6. (a) i. Let $f(x)$ be a continuous, non-negative function on $[a, b]$. By considering Riemann sums, derive the formula

$$V = \pi \int_a^b [f(x)]^2\, dx$$

for the volume of the solid generated by revolving the region under the curve $y = f(x)$ about the x-axis.

[20 Marks]

Continued...

ii. Using the above result, find the volume of the solid generated by revolving the region under $y = \sin x$ on the interval $[0, \pi]$ is revolved about the x-axis.

[15 Marks]

(b) i. Let $f(x)$ be a continuously differentiable function on $[a, b]$. Derive the formula for the arc length L of the curve $y = f(x)$, from $x = a$ to $x = b$, and show that

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

[20 Marks]

ii. Using the above result, find the length of the curve

$$y = \frac{1}{2}\left(e^x + e^{-x}\right)$$

from $x = 0$ to $x = 3$.

[20 Marks]

(c) The lateral (side) surface area of a cone with base radius r and height h is given by the product of the semiperimeter of the base and the slant height, that is,

$$A = (\pi r)\sqrt{r^2 + h^2}.$$

Verify this formula by finding the surface area generated when the line segment

$$y = \frac{r}{h}x, \quad 0 \le x \le h$$

is revolved about the x-axis.

[25 Marks]
[Total 100 Marks]

*****End of the Examination Paper*****