



University of Kelaniya - Sri Lanka
Center for Distance & Continuing Education
Bachelor of Science(General) External
First year second semester examination - 2023 (2025 June)
(New Syllabus)
Faculty of Science
Pure Mathematics
PMAT 17543 - Theory of Calculus

No. of Questions: Six(06) No. of Pages: Three(03) Time: Two and Half($2\frac{1}{2}$) hrs
Answer Five(05) Questions Only

1. (a) Let S be a nonempty subset of $\mathbb{R}$, state the definitions of upper bound of S and lower bound of S .

[20 marks]

- (b) For each sets below determine whether each set is bounded above, bounded below, or both. If it is bounded above (below) then find the supremum (infimum). Justify all your answers.

(i) $\left\{ \frac{3n-1}{n}, n \in \mathbb{N} \right\}$
(ii) $\left\{ \frac{n+(-1)^n n}{n+1}, n \in \mathbb{N} \right\}$
(iii) $\left\{ 1 + \frac{1+(-1)^n}{n}, n \in \mathbb{N} \right\}$

[60 marks]

- (c) If x and y are two real numbers such that $x < y$, then prove that, there exists a rational number r such that $x < r < y$.

[20 marks]

2. (a) If $f(x) = \sqrt{5-x}$ and $g(x) = \sqrt{x-1}$ find the composite functions (i) $f \circ g$, (ii) $g \circ f$, (iii) $f \circ f$ and (iv) $g \circ g$ and their domains.

[40 marks]

- (b) (i) If $f(x) = \frac{|x+4|}{2x+8}$, Find $\lim_{x \rightarrow -4^-} f(x)$, $\lim_{x \rightarrow -4^+} f(x)$ and explain whether the limit $\lim_{x \rightarrow -4} f(x)$ exists?

[15 marks]

Continued.

(ii) Evaluate the following limits if they exist

$$\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^4 + 3x^2 - 4}$$

$$\lim_{u \rightarrow 0} \left(\frac{\sqrt{1+u} - \sqrt{1-u}}{u} \right)$$

[20 marks]

(c) States the Squeezes Theorem in the usual notation.

(i) Prove that

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin(\pi/x) = 0$$

(ii) Prove that

$$\lim_{x \rightarrow 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0$$

[25 marks]

3. (a) State the Intermediate Value Theorem.

[10 marks]

(b) Prove, without graphing, that the graph of the function $y = x^2 - 3 + 1/x$ has at least two x -intercepts in the $(0, 2)$ interval.

[35 marks]

(c) For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

[30 marks]

(d) If f is differentiable at a , then prove that f is continuous at a .

[25 marks]

4. (a) In the usual notation, state the definition of derivative of a function f . Use the definition of derivative to find the second derivative of $f(x) = x^3(x+3)$. Justify your answer using relevant derivative rules.

[40 marks]

(b) Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.

(i) Find $g'(0)$ and $g''(0)$ in terms of c .

(ii) In terms of k , find an equation of the tangent line to the graph of h at the point where $x = 0$.

[30 marks]

(c) Use implicit differentiation to find the derivative of the curve $\tan(x+y) + \sec(x-y) = 2$ with respect to x , at the point $(\pi/8, \pi/8)$.

[30 marks]

Continued.

5. (a) Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a . If $f(a) = g(a) = 0$, then show that

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f(x)}{g(x)},$$

by using the definition of a derivative.

[25 marks]

- (b) Find the limits of the followings. Use L'Hospital's Rule where appropriate.

(i) $\lim_{x \rightarrow \infty} \frac{x + x^2}{1 - 2x^2}$

(ii) $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{e^x - x - 1}$

[30 marks]

- (c) Show that the definite integral $\int_0^4 (x - x^2) dx$ is equal to $\lim_{n \rightarrow \infty} R_n$, where

$$R_n = \frac{4}{n} \sum_{i=1}^n \left[\frac{4i}{n} - \frac{16i^2}{n^2} \right]$$

and then evaluate the limit.

Hint: $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

[45 marks]

6. (a) Find the volume of the solid obtained by rotating the region bounded by $8y = x^2$, $y = \sqrt{x}$, about the y -axis. Sketch the solid and a typical disk or washer.

[30 marks]

- (b) Suppose the region bounded by $y = x^2$, $y = 8\sqrt{x}$ is rotated about the y -axis. Sketch the region and find the volume of the resulting solid using,

(i) Cylindrical shells method and sketch the solid.

(ii) Washer method and sketch the solid.

[50 marks]

- (c) Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.

Hint: $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$, where L is the arc length.

[20 marks]

END