UNIVERSITY OF KELANIYA SRI LANKA
Center for Distance and Continuing Education

Bachelor of Science (External) First Year First Semester Examination
2024 – December
Faculty of Science

APPLIED MATHEMATICS
AMAT 16522- Mechanics I

No. of Questions: Five (06) No. of Pages: Three (03) Time: Two (02) hours

Answer Four (04) Questions only.

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1. (a) Define an inertial frame in Newtonian Mechanics.

   (b) Consider two reference frames S and S' which were coincide at $t = 0$. Suppose that the frame S' is moving with the velocity v in the direction of the positive $x-$axis relative to the frame S. In the usual notation, obtain the Lorentz transformation equations between the two reference frames S and S' in the form

$$x' = \gamma\,(x - vt)\,,$$
$$y' = y,$$
$$z' = z,$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

   Hence, show that the length contraction formula takes the form

$$L_0 = \gamma L,$$

   where L_0 is the length of the object placed in the frame S' and L is the length of the object measured by an observer in the frame S.

   (c) A stick with length 1.5m(meters) is placed in an aircraft which is moving with a speed $\frac{c}{3}$ with respect to an stationary observer who is on the earth. What is the length of the stick as seen by the observer on earth ?

2. (a) In the usual notation, using polar coordinates, obtain the radial and transverse components of acceleration $\left(\ddot{r} - r\dot{\theta}^2\right)$ and $\frac{1}{r}\frac{d(r^2\dot{\theta})}{dt}$ of a particle moving in a plane, respectively.

(b) A particle moves in such a way that its radial and transverse velocities are always $2\lambda a\theta$ and λr. Show that its acceleration in these two directions are given by $\lambda^2(2a-r)$ and $4a\lambda^2\theta$ and its path is the curve described by $r = a\theta^2 + c$. Here λ, a and c are some constants.

(c) Consider two particles, each with mass M, moving in opposite directions at velocities v_1 and v_2, respectively. Suppose E denotes the total kinetic energy and P denotes the total momentum. Then show that

$$E - \frac{P^2}{4M} = \frac{M}{4}(v_1 + v_2)^2.$$

3. (a) In the usual notation, obtain the formulae

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}, \text{ and } \dot{\theta} = hu^2,$$

for a motion of a particle that describes an orbit, under the central force F per unit mass in the usual notation, where $u = \frac{1}{r}$ and h is a constant.

(b) A particle describes the orbit $r = a\sin n\theta$ under the central force F to the pole, where a and n are constants. Show that

$$F = h^2\left(\frac{1-n^2}{r^3} + \frac{2a^2n^2}{r^5}\right).$$

(c) Prove that the motion of a particle of mass m under a central force towards a fixed point is always a plane curve.

4. (a) Consider the motion described by the **question 3(a) above**. Show that

i. $\dot{r} = -h\dfrac{du}{d\theta}$,

ii. $v^2 = \dot{r}^2 + (r\dot{\theta})^2 = h^2\left[u^2 + \left(\dfrac{du}{d\theta}\right)^2\right]$, where v is the velocity of the particle.

(b) A particle of unit mass projected from an apse at a distance $(a+b)$ with velocity $\dfrac{\sqrt{\mu}}{(a+b)}$ moves with a central force $\mu\left(3au^4 - 2(a^2-b^2)u^5\right)$; where $a > b$. Prove that in the subsequent motion, the velocity of the particle is given by

$$v^2 = \mu\left(2au^3 - (a^2-b^2)u^4\right).$$

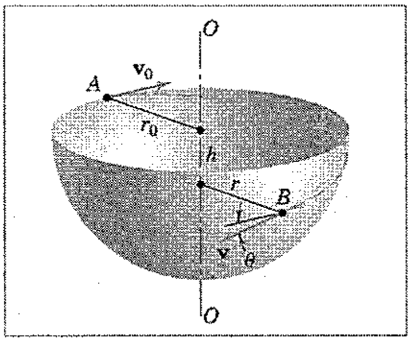
(c) Hence, show that the total energy required for the orbit of the particle is

$$\frac{\mu}{2r^4}\left(2ar - (a^2-b^2)\right) + \int_{(a+b)}^{r} \frac{\mu}{r^5}\left(3ar - 2(a^2-b^2)\right) dr$$

(You can assume that the system is conservative.)

5. (a) A particle P of mass m is moving along a curve in a space. If the particle is located by its position vector $\mathbf{r}$ with respect to a origin O of a fixed coordinate system, define the angular momentum of P.

(b) A particle of a small mass is given an initial velocity v_0 tangent to the horizontal rim of a smooth hemispherical bowl at a radius r_0 from the vertical center-line, (as shown in the figure). As the particle slides past point B, a distance h below A and a distance r from the vertical center-line, its velocity v makes an angle θ with the horizontal tangent to the bowl through B. Determine the angle θ in terms of r_0, v_0 and h.



(c) A particle describes an ellipse of eccentricity e about a center of force at a focus. Prove in the usual notation

$$v^2 = \frac{h^2}{l}\left(\frac{2}{r} - \frac{1}{a}\right),$$

where $a(1-e^2) = l$.

Hence, deduce that for this case, the velocity is not exceeding $\sqrt{\frac{2h^2}{lr}}$.

6. (a) Assume in the usual notation $\frac{d\underline{A}}{dt} = \frac{\partial \underline{A}}{\partial t} + \underline{\omega} \times \underline{A}$ for a vector $\underline{A}$ with respect to two references of frames, one is rotating relative to the other (here $\frac{\partial \underline{A}}{\partial t}$ is the derivative of A with respect to the rotating frame). Then prove that

$$\frac{d^2\underline{r}}{dt^2} = \frac{\partial^2 \underline{r}}{\partial t^2} + \frac{\partial \underline{\omega}}{\partial t} \times \underline{r} + 2\underline{\omega} \times \frac{\partial \underline{r}}{\partial t} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

where $\underline{r}$ is the position vector of the particle with respect to the origin on the surface of the earth and $\underline{\omega}$ is the angular velocity of the earth's diurnal rotation.

(b) An xyz coordinate system is rotating with respect to an XYZ coordinate system with the same origin and is assumed to be fixed in space. The angular velocity of the xyz system relative to the fixed system is given by $\underline{\omega} = 2\underline{i} - t\underline{j} + 6t\underline{k}$. The position vector of a particle at time t as observed in xyz system is given by $\underline{r} = t\underline{i} - t\underline{j} + \underline{k}$. Find the acceleration of the particle as seen by an observer in the fixed system.

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