University of Kelaniya - Sri Lanka
Center for Distance & Continuing Education
Bachelor of Science(External) First Year First Semester- 2023
2024-December
Faculty of Science

Applied Mathematics
AMAT 16513 - Vector Analysis

No.of Questions: Six(06) **No.of Pages: Three(03)** **Time: Two & half($2\frac{1}{2}$)hrs**

Answer **Five(05)** Questions Only

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1. (a) Find the vector projection of $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ onto $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$.

   (b) Check whether the given vectors $\vec{u} = \hat{i}+5\hat{j}-2\hat{k}$, $\vec{v} = 3\hat{i}-\hat{j}$ and $\vec{w} = 5\hat{i}+9\hat{j}+4\hat{k}$ are coplanar or not. Justify your answer.

   (c) Find the angle between two planes $3x + 4y = 0$ and $2x + y - 2z = 5$. (Do not need to simplify the answer.)

   (d) Let two non-zero vectors $\vec{v_1}$ and $\vec{v_2}$ have magnitudes w and z, respectively. Given that $x = |\vec{v_1} \times \vec{v_2}|$ and $y = \vec{v_1} \cdot \vec{v_2}$. Show that $x^2 + y^2 = w^2 z^2$.

   (e) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.

2. (a) A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.

   (i) Determine the velocity and acceleration of the particle at any time.

   (i) Find the magnitude of the velocity and acceleration at $t = 0$.

   (b) A curve C is defined by parametric equation $x = x(s)$, $y = y(s)$, $z = z(s)$, where s is length of C measured from fixed point on C. If $\vec{r}$ is a position vector of any point on C, show that $\frac{d\vec{r}}{ds}$ is a unit vector tangent to C.

   [Hint:$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$]

   (c) Show that $\vec{A} \cdot \frac{d\vec{A}}{dt} = A\frac{dA}{dt}$ where $\vec{A}$ is a vector and A is the magnitude of the vector $\vec{A}$.

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3. (a) Let $\vec{a}$ be a constant vector given by $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\hat{k}$ and $\vec{r}$ be a position vector of any point p with respect to the origin, given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Show that the followings in the usual notation.

(i) $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$ (ii) $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$

(b) Determine the arc length of the curve $\vec{r}(t) = e^t \cos t\,\hat{i} + e^t \sin t\,\hat{j}$ between $t = 0$ and $t = 1$.

(c) The position vector of moving particle is $\vec{r}(t) = \cos t\,(\hat{i} - \hat{j}) + \sin t\,(\hat{i} + \hat{j}) + \frac{t}{2}\,\hat{k}$

(i) Find a unit tangent vector to the path of the particle, in the direction of motion.

(ii) Show that the curve traversed by the particle has constant curvature κ and find its value.

(d) Show that $\dfrac{d\vec{N}}{ds} + \kappa\vec{T}$ is perpendicular to both $\vec{T}$ and $\vec{N}$, where $\vec{T}$ is the unit tangent vector and $\vec{N}$ is the unit normal vector.

Hint: Frenet- Serret Formula $\dfrac{d\vec{N}}{ds} = -\kappa\vec{T} + \tau\vec{B}$ where $\vec{B}$ is the binormal vector. Also κ and τ are curvature and torsion respectively and consider them as constants.

4. (a) Find constants a, b, c so that

$$\vec{v} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

is irrotational.

(b) Evaluate the surface integral $\iint_S (x+y^2)dS$ is the part of the plane $x+y+z = 2$ in the first octant and has upward orientation.

(c) (i) State the Stokes' Theorem.

(ii) Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2 z\,\hat{i} - yz\hat{k}$ and C is the curve intersection of the plane $z = 3$, and the cylinder $x^2 + y^2 = 25$, oriented counterclockwise.

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5. (a) Evaluate $\int_C \sqrt{1+y^2}\, ds$ where C is the curve $y = e^x$ from $(0, 1)$ to $(1, e)$.

(b) Given the vector field $\vec{F} =< 2xe^y + 1, x^2e^y >$

(i) Show that $\vec{F}$ is a conservative vector field.

(ii) Find a potential function $f(x, y)$ such that $\vec{F} = \nabla f$.

(iii) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the curve $\vec{r}(t) = 1 - 2t + 3t^{2024}\, \hat{\imath} + \dfrac{(1-t)}{\sqrt{4+t}}\, \hat{\jmath}$ with $0 \leq t \leq 1$.

(c) Evaluate $\oint \vec{F} \cdot d\vec{r}$, where $\vec{F} =< x\tan^{-1} x\sqrt{\ln(1+x)}\ ,\ \frac{5}{2}x^2 + \sin y^3 >$ and C is the boundary of the region enclosed by the curves $y = x^3$, $x = 2$ and $y = 0$ with counterclockwise orientation.

6. (a) Prove

(i) $\nabla\times(\nabla\phi) = 0$ (Curl grad $\phi = 0$) (ii) $\nabla\cdot(\nabla\times\vec{A}) = 0$ (div Curl $\vec{A} = 0$)

where $\vec{A} = A_1\,\hat{\imath} + A_2\,\hat{\jmath} + A_3\hat{k}$ is a vector, ϕ is a scalar function and assume both ϕ and $\vec{A}$ have continuous second partial derivatives.

(b) Evaluate $\nabla \cdot (\vec{A} \times \vec{r})$ if $\nabla \times \vec{A} = 0$, where $\vec{A}$ is a vector given by $\vec{A} = A_1\,\hat{\imath} + A_2\,\hat{\jmath} + A_3\,\hat{k}$ and $\vec{r}$ is a vector given by $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

(c) (i) State the Divergence Theorem.

(ii) Use Divergence Theorem to calculate the surface integral of $\vec{F} = x^2y\,\hat{\imath} + xe^z\,\hat{\jmath} + z^2\,\hat{k}$ across the surface of the box with vertices $(\pm1, \pm2, \pm3)$.

— End of Examination —