UNIVERSITY OF KELANIYA- SRI LANKA

Centre for Distance & Continuing Education

Bachelor of Science (General) External

First Year First Semester Examination- 2019

(New Syllabus) 2022 August

Faculty of Science

APPLIED MATHEMATICS

AMAT 16513- Vector Analysis

No. of Questions: Seven (07) Time: Two and a half (02 1/2)

No. of Pages: Four (04)

Answer Five (05) Questions only

---

1. (a) Suppose $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors. Prove that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$ implies that $x = y = z = 0$.

   (b) Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of the points A and B with respect to the origin $\mathbf{O}$. Find the equation of the straight line that passes through the given points A and B.

   (c) Suppose $\mathbf{a}$ and $\mathbf{b}$ are known vectors with $\mathbf{a} \neq \mathbf{0}$. Prove that $\mathbf{a} \times \mathbf{y} = \mathbf{b}$ has a solution if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

   (d) Let $\cos\alpha, \cos\beta$ and $\cos\gamma$ be the direction cosines of the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. In the usual notation, show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

2. (a) Suppose $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find a vector of magnitude 5 and perpendicular to both **A** and **B**.

(b) Suppose $\mathbf{A} = 3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ are the position vectors of the points P and Q, respectively.

i. Find an equation for the plane passing through Q and perpendicular to the line PQ.

ii. Find the distance from the point $(-1, 1, 1)$ to the plane.

(c) Find the volume of the parallelepiped with sides **A**, **B** and **C** in the usual notation. Hence show that **A**, **B** and **C** are coplanar if and only if $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$.

3. (a) Suppose a particle moves along the curve $x = 2\sin 3t, y = 2\cos 3t, z = 8t$ at any time $t > 0$.

i. Find the velocity and acceleration of the particle.

ii. Find the magnitude of the velocity and acceleration at any time t.

(b) Find a unit tangent vector to any point on the curve $x = a\cos\omega t, y = a\sin\omega t, z = bt$ where a, b and ω are constants.

(c) Let **A** and **B** be differential vector functions of s. Prove the following identities.

i. $\frac{d}{ds}\left(\mathbf{A} \cdot \frac{d\mathbf{B}}{ds} - \frac{d\mathbf{A}}{ds} \cdot \mathbf{B}\right) = \mathbf{A} \cdot \frac{d^2\mathbf{B}}{ds^2} - \frac{d^2\mathbf{A}}{ds^2} \cdot \mathbf{B}$.

ii. $\frac{d}{ds}\left(\mathbf{A} \times \frac{d\mathbf{B}}{ds} - \frac{d\mathbf{A}}{ds} \times \mathbf{B}\right) = \mathbf{A} \times \frac{d^2\mathbf{B}}{ds^2} - \frac{d^2\mathbf{A}}{ds^2} \times \mathbf{B}$.

(d) Suppose **A** has a constant magnitude. Show that $\mathbf{A} \cdot d\mathbf{A}/dt = 0$ and that **A** and $d\mathbf{A}/dt$ are perpendicular provided that $|d\mathbf{A}/dt| \neq 0$.

4. (a) In the usual notation prove the following identities:

   i. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$.

   ii. $\nabla \cdot (\phi \mathbf{A}) = (\nabla \phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A})$.

   iii. $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = 0$.

   (b) Find a unit normal (outward drawn) to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point $(3, 1, -4)$.

   (c) Find the directional derivative of the function $4e^{2x-y+z}$ at the point $(1, 1, -1)$ in a direction toward the point $(-3, 5, 6)$.

   (d) Prove that the vector $\mathbf{A} = 3y^4z^2\mathbf{i} + 4x^3z^2\mathbf{j} - 3x^2y^2\mathbf{k}$ is solenoidal.

5. (a) Suppose $\mathbf{v} = \omega \times \mathbf{r}$ where ω is a constant vector. Prove, in the usual notation, that

$$\omega = \frac{1}{2}(\nabla \times \mathbf{V}).$$

   (b) Suppose $\nabla \times \mathbf{A} = \mathbf{0}$. Prove in the usual notation that

$$\nabla \cdot (\mathbf{A} \times \mathbf{r}) = \mathbf{r} \cdot (\nabla \times \mathbf{A}).$$

   (c) Suppose $\mathbf{A}$ and $\mathbf{B}$ are irrotational. Prove that $\mathbf{A} \times \mathbf{B}$ is solenoidal.

   (d) i. Find constants a, b and c so that $\mathbf{V} = (4x - 2y + az)\mathbf{i} + (bx - 2y + 5z)\mathbf{j} + (4x + cy + 5z)\mathbf{k}$ is irrotational.

   ii. Show that $\mathbf{V}$ can be expressed as the gradient of a scalar function. Find that scalar function.

6. (a) State the Divergence Theorem.

   (b) Suppose S is any closed surface enclosing a volume V, $\mathbf{n}$ is the positive normal to S and $\mathbf{A} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$. Prove that

$$\iint_S \mathbf{A} \cdot \mathbf{n} dS = (a + b + c)\ V$$

   (c) Evaluate $\iint_S \mathbf{r} \cdot \mathbf{n}\ dS$ in the usual notation, where S is a closed surface and $\mathbf{n}$ is the positive normal to S.

   (d) Use part (c) above to evaluate $\iint_S \mathbf{r} \cdot \mathbf{n}\ dS$, where

      i. S is the sphere of radius 2 with center at (0,0,0),

      ii. S is the surface of the cube bounded by $x = -1, y = -1, z = -1, x = 1, y = 1$ and $z = 1$

7. (a) State the Stokes' Theorem.

   (b) Evaluate $\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\ dS$, where $\mathbf{A} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$ and S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 16$.

   (c) Prove that

$$\oint d\mathbf{r} \times \mathbf{B} = \iint_S (\mathbf{n} \times \nabla) \times \mathbf{B}\ dS,$$

   in the usual notation.